4 Exercise Sheet 4

Exercise 4.1 (Rademacher's theorem)**.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz continuous function. Prove that f is differentiable $\mathscr{L}^d$-almost everywhere.*

Hint:

1. Using Riesz theorem, prove that there exists a *weak gradient* of f, namely there exists an L^∞-function $\tilde{\nabla} f = (\tilde{\partial}_1 f, \ldots, \tilde{\partial}_d f) : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} f \, \partial_i g \, dx = - \int_{\mathbb{R}^d} \tilde{\partial}_i f \, g \, dx$$

   for all $g \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ and for all $i = 1, \ldots, d$. Here with $\partial_i g$ we denote the classical derivative of g with respect to the coordinate x_i.

   To do this, use that $\partial_i g$ is a limit of incremental ratios.

2. Let $x_0 \in \mathbb{R}^d$ be a Lebesgue point for $\tilde{\nabla} f$ and call $A : \mathbb{R}^d \to \mathbb{R}^d$ the linear map such that

$$\fint_{B_r(x_0)} |\tilde{\nabla} f - A| \, dx \to 0$$

   as $r \to 0$, which exists by definition of Lebesgue point.

3. Show that the sequence of functions $f_r(y) = (f(x_0 + ry) - f(x_0))/r$ admits a uniformly convergent subsequence to a function f_0 in $\overline{B(0,1)}$.

4. Prove that the weak gradients $\tilde{\nabla} f_r$ converge to A in $L^1(B(0,1), \mathbb{R}^d)$ as $r \to 0$.

5. Show that $f_0 = A$ in $B(0,1)$ and deduce that A is the classical gradient of f at x_0.

Exercise 4.2. *Consider n red points $P_1, \ldots, P_n$ and n blue points $Q_1, \ldots, Q_n$ on the plane. Assume that these $2n$ points are distinct and there are no 3 collinear points.*

Show that it is possible to connect each red point to a distinct blue point with a segment in such a way that these segments do not intersect each other. Namely, there exists a permutation $\sigma \in S_n$ such that the segment $P_i Q_{\sigma(i)}$ does not intersect the segment $P_j Q_{\sigma(j)}$ for any $i \neq j$.