5 Exercise Sheet 5

Exercise 5.1. *Solve again exercises 3.1 and 3.2 using the sufficient condition for the optimality of a transport map with respect to the quadratic cost.*

Exercise 5.2. *Show that the nondecreasing map $T : \mathbb{R} \to \mathbb{R}$ constructed in exercise 1.1 is optimal with respect to the quadratic cost.*

Exercise 5.3. *Find the optimal transport map for the quadratic cost $c(x, y) = \frac{1}{2}|x - y|^2$ between $\mu = f \cdot \mathscr{L}^2$ and $\nu = g \cdot \mathscr{L}^2$ in $\mathbb{R}^2$, where $f(x) = \frac{1}{\pi}\chi_{B(0,1)}(x)$ and $g(x) = \frac{1}{8\pi}(4 - |x|^2)\chi_{B(0,2)}(x)$.*

Exercise 5.4. *Let $S : \mathbb{R}^d \to \mathbb{R}^d$ be given by $S(x) = -x$. Characterize the probabilities $\mu \in \mathcal{P}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} |x|^2 d\mu < +\infty$ such that S is an optimal transport map between μ and $S_{\#}\mu$ for the quadratic cost.*