6 Exercise Sheet 6

This whole exercise sheet is devoted to the disintegration theorem. The first exercise shows the intuition behind the disintegration, whereas the other three exercises are increasingly general statements of the theorem. The difficulty of the proof is contained entirely in exercise 6.2.

Exercise 6.1. *Let $\mu \in \mathscr{M}(\mathbb{R}^2)$ be a finite measure on $\mathbb{R}^2$ that is absolutely continuous with respect to the Lebesgue measure with density $\rho : \mathbb{R}^2 \to \mathbb{R}$. Let $\nu \in \mathscr{M}(\mathbb{R})$ be the measure with density $\eta(x) = \int_{\mathbb{R}} \rho(x, y)dy$. For any $x \in \mathbb{R}$ such that $\eta(x) \neq 0$, let μ_x be the measure with density $\rho_x(y) = \frac{\rho(x,y)}{\eta(x)}$. If $\eta(x) = 0$, then simply set $\mu_x = 0$.*

Show that for any $g \in L^1(\mu)$ it holds

$$\int_{\mathbb{R}^2} g(x, y)d\mu(x, y) = \int_{\mathbb{R}} \int_{\mathbb{R}} g(x, y)d\mu_x(y)d\nu(x).$$

Exercise 6.2 (Disintegration for product of compact spaces)**.** *Let X, Y be two compact spaces and let $\mu \in \mathscr{M}(X \times Y)$ be a finite measure on the product $X \times Y$. Let us denote $\nu = (\pi_1)_{\#}\mu$ where $\pi_1 : X \times Y \to X$ is the projection on the first coordinate. Prove that there exists a family of probabilities $(\mu_x)_{x\in X} \subseteq \mathscr{P}(Y)$ such that:*

(a) For any Borel set $E \in \mathscr{B}(Y)$ the map $x \mapsto \mu_x(E)$ is Borel[4]*.*

(b) For any $g \in L^1(\mu)$ it holds

$$\int_{X\times Y} g(x, y)d\mu(x, y) = \int_X \int_Y g(x, y)d\mu_x(y)d\nu(x).$$

Hint:

1. Given $\psi \in C^0(Y)$, consider the map $A_\psi : L^1(X, \nu) \to \mathbb{R}$ given by the formula $A_\psi(\phi) := \int_{X\times Y} \phi(x)\psi(y)d\mu(x, y)$. Prove that the said map is linear continuous and therefore $A_\psi \in L^\infty(X, \nu)$.
2. Fix a countable dense subset $S \subseteq C^0(Y)$. Prove that for ν-almost every $x \in X$ the map $\mu_x : S \to \mathbb{R}$ given by $\mu_x(\psi) = A_\psi(x)$ is linear continuous and therefore $\mu_x \in \mathscr{P}(Y)$. Show that the said family $(\mu_x)_{x\in X}$ satisfies *(a)*.
3. Show that *(b)* holds when $g \in L^1(X, \nu) \times S$. Show that this implies that it holds also when $g \in L^1(X, \nu) \times C^0(Y)$. Finally show that this implies *(b)* for any $g \in L^1(\mu)$.

Exercise 6.3 (Disintegration for product of Polish spaces)**.** *Show the statement of the previous exercise when X and Y are Polish spaces, i.e. they are complete and separable.*

Hint:

Use Prokhorov's theorem to find a suitable exhaustion in compact sets that allows to apply the previous exercise.

Exercise 6.4 (Disintegration for fibers of a map)**.** *Let X, Y be two Polish spaces, let $f : Y \to X$ be a Borel map and let $\mu \in \mathscr{M}(Y)$ be a finite measure on Y. Let us denote $\nu := f_{\#}\mu$. Show that there exists a family of probabilities $(\mu_x)_{x\in X} \subseteq \mathscr{P}(Y)$ such that:*

[4] This fact is necessary to give a meaning to the integral in the *real* statement of the theorem.

- *For any Borel set* $E \in \mathscr{B}(Y)$ *the map* $x \mapsto \mu_x(E)$ *is Borel.*
- *For* ν*-almost every* $x \in X$ *the measure* μ_x *is supported on the fiber* $f^{-1}(x)$*.*
- *For any* $g \in L^1(\mu)$ *it holds*

$$\int_Y g(y)d\mu(y) = \int_X \int_{f^{-1}(x)} g(y)d\mu_x(y)d\nu(x).$$

Hint:
Apply the previous exercise on the measure $(f \times \mathrm{id})_{\#}\mu$.