D-MATH | Functional analysis I | Fall 2018
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Exercise sheet 0

This exercise sheet treats topics from point set topology which will be useful in the course and which you might already know. In particular, it contains a proof of the general case of Tychonoff's Theorem using ultrafilters.

1. (**Initial topologies and examples**). Let X be a set, let $\mathcal{I}$ be an index set and assume that for any $\iota \in \mathcal{I}$ a topological space X_ι and a map $f_\iota : X \to X_\iota$ is given.

a) Show that there is a smallest topology on X containing all sets of the form $f_\iota^{-1}(U_\iota)$ where $U_\iota \subset X_\iota$ is open. This topology on X is called the *initial topology* induced by the maps f_ι for $\iota \in \mathcal{I}$ and is the smallest topology which makes all maps f_ι continuous.

We now illustrate this notion in an example.

b) Let $X = \prod_{\iota \in \mathcal{I}} X_\iota$. Then the initial topology induced by the projections $X \to X_\iota$ is called the *product topology* on X. Assume that $\mathcal{I}$ is countable and that X_ι is metrizable[1] for any $\iota \in \mathcal{I}$. Show that X is metrizable.

HINT: The proof is contained in[2] Lemma A.17.

2. (**Urysohn's lemma**). Let X be a topological space. We say that X is *normal* if for any two disjoint closed sets $A, B \subset X$ there are disjoint open sets $U_A, U_B \subset X$ with $U_A \supset A$ and $U_B \supset B$. If X is normal, Urysohn's lemma states that for any two disjoint closed subsets A, B of X there is a continuous function $f : X \to [0,1]$ with $f|_A \equiv 0$ and $f|_B \equiv 1$.

Assume that X is metrizable and let d be a metric on X inducing the topology. For any non-empty subset $A \subset X$ and a point $x \in X$ define

$$\mathrm{d}(x, A) = \inf\{\mathrm{d}(x,a) : a \in A\}.$$

[1] That is, there is a metric on X_ι which induces the given topology on X_ι.

[2] If not explicitly stated otherwise, we make reference to the book of Einsiedler and Ward *Functional analysis, Spectral theory, and Applications*.

For $A, B \subset X$ closed, non-empty and disjoint we consider the function

$$f : x \in X \mapsto \frac{\mathrm{d}(x, A)}{\mathrm{d}(x, A) + \mathrm{d}(x, B)}.$$

Show that it satisifes all the properties required in Urysohn's lemma. Deduce that any metrizable topological space is normal.

REMARK: You can read up on the proof of the general case of Urysohn's lemma in Section A.5.

In the following three exercises we would like to discuss *Tychonoff's Theorem* which states the following. Let $(X_\iota)_{\iota\in\mathcal{I}}$ be a collection of topological spaces. Then the product space $\prod_{\iota\in\mathcal{I}} X_\iota$ is compact if and only if X_ι is compact for any $\iota \in \mathcal{I}$.

3. (**A special case**) Show that a countable product of metrizable compact topological spaces is also compact.

HINT: You may use that a metrizable space is compact if and only if it is sequentially compact.

In the general situation one useful replacement for sequences is given by filters[3]. Recall that a *filter* on a topological space X is a subset $\mathcal{F} \subset \mathcal{P}(X)$ with the following properties: (i) $\emptyset \notin \mathcal{F}$, (ii) $F_1, F_2 \in \mathcal{F} \implies F_1 \cap F_2 \in \mathcal{F}$ and (iii) $F \in \mathcal{F}, B \supset F \implies B \in \mathcal{F}$. For instance, the set of neighborhoods $\mathcal{U}_x$ of a point $x \in X$ is a filter. We say that a filter $\mathcal{F}$ on X *converges* to $x \in X$ if $\mathcal{U}_x \subset \mathcal{F}$.

4. (**More about filters**) Let X be a topological space.

a) Let $(x_n)_n$ be a sequence in X. Show that

$$\mathcal{F}_{\text{tail}} = \{F \subset X : x_n \in F \text{ for all but finitely many } n\}$$

is a filter on X and that $\mathcal{F}_{\text{tail}}$ converges to $x \in X$ if and only if $(x_n)_n$ converges to x. Thus, convergence of filters generalizes convergence of sequences.

A filter $\mathcal{F}$ on X is an *ultrafilter* if it is maximal amongst all filter on X with respect to inclusion.

b) Show that a filter $\mathcal{F}$ on X is an ultrafilter if and only if $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$ for any subset A of X. Conclude from this that an ultrafilter $\mathcal{F}$ converges to $x \in X$ if $U \cap F \neq \emptyset$ for any $F \in \mathcal{F}$ and any neighborhood U of x.

[3] A very complete reference on filters (and nets) is P. Clark's short note *Convergence*. Here, we will only introduce as much as is needed to understand our proof of Tychonoff's Theorem.

c) Let $f : X \to Y$ be a continuous map between topological spaces and let $\mathcal{F}$ be an (ultra-)filter on X. Show that $f(\mathcal{F}) = \{B \subset Y : f(F) \subset B \text{ for some } F \in \mathcal{F}\}$ is an (ultra-)filter on Y and that $f(\mathcal{F})$ converges to $f(x)$ if $\mathcal{F}$ converges to $x \in X$.

d) Show that any filter is contained in an ultrafilter.

HINT: Use Zorn's lemma.

5. (**Proof of Tychonoff's Theorem**)

a) Let X be a topological space. Show that X is compact if and only if every ultrafilter on X converges.

HINT: Assuming that X is compact find a point x in the intersection $\bigcap_{F \in \mathcal{F}} \overline{F}$ whenever $\mathcal{F}$ is an ultrafilter on X. Conversely, if $\mathcal{A}$ is a family of closed subsets with $A_1 \cap \ldots \cap A_n \neq \emptyset$ for any $A_1, \ldots, A_n \in \mathcal{A}$ consider the filter of subsets of X which contain a finite intersection of elements of $\mathcal{A}$.

b) Prove Tychonoff's Theorem.

NOTE: The vorxn-system is inactive for this exercise sheet and will be used from Exercise sheet 1 onward. If you wish to present an exercise nevertheless, write an e-mail to your assistant who will be able to give you one point for a presentation in the exercise class.