D-MATH
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Exercise sheet 1

1. (**Products**). Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed vector spaces. Show that

$$\|(v,w)\|_{V\times W} = \max\{\|v\|_V, \|w\|_W\}$$

for $(v,w) \in V \times W$ defines a norm $\|\cdot\|_{V\times W}$ on the product space $V \times W$.

2. (**Norm equivalence via convergence**). Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$ and let $\|\cdot\|$, $\|\cdot\|'$ be two norms on V. Show that $\|\cdot\|$ and $\|\cdot\|'$ are equivalent if and only if any sequence $(v_n)_n$ in V which converges to some $v \in V$ with respect to $\|\cdot\|$ also converges to v with respect to $\|\cdot\|'$ and conversely.

3. (**Example of inequivalent norms**). Let $C^1([0,1])$ be the space of continuous functions $f : [0,1] \to \mathbb{R}$ which have a continuous derivative f'. The C^1-norm $\|\cdot\|_{C^1([0,1])}$ on $C^1([0,1])$ is given by

$$\|f\|_{C^1([0,1])} = \max\{\|f\|_\infty, \|f'\|_\infty\}$$

for $f \in C^1([0,1])$.

a) Show that the supremum norm $\|\cdot\|_\infty$ on $C^1([0,1])$ is inequivalent to the C^1-norm.

b) Verify that $\|f\|_0 = |f(0)| + \|f'\|_\infty$ for $f \in C^1([0,1])$ defines a norm on $C^1([0,1])$ which is equivalent to $\|\cdot\|_{C^1([0,1])}$.

4. (**ℓ^p-spaces**). We define for any $p \geq 1$ the set of complex-valued sequences

$$\ell^p(\mathbb{N}) = \left\{ x = (x_n)_n : \|x\|_p = \Big(\sum_{n=1}^\infty |x_n|^p \Big)^{\frac{1}{p}} < \infty \right\}.$$

In this exercise we would like to show that $\ell^p(\mathbb{N})$ is a vector space and that $\|\cdot\|_p$ defines a norm on it.

a) Prove Young's inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ where $q \geq 1$ satisfies $1 = \frac{1}{p} + \frac{1}{q}$ and where $a, b \geq 0$.

b) Prove Hölder's inequality

$$\sum_{n=1}^{\infty} |x_n||y_n| \leq \|x\|_p \cdot \|y\|_q$$

for any $x \in \ell^p(\mathbb{N})$ and $y \in \ell^q(\mathbb{N})$.

c) Use b) to deduce the desired statement.

HINT: To prove the triangle inequality, start with an expression of the kind $\sum_{n=1}^{N} |x_n + y_n|^p$ and apply Hölder's inequality.

5. (**Sums of subspaces**). This exercise exhibits a "strange" phenomenon in a infinite-dimensional normed vector space, which cannot be seen in finite-dimensional vector spaces. Consider the normed vector space $V = \ell^1(\mathbb{N}) \times \ell^1(\mathbb{N})$ (cf. Exercises 1 and 4) and the subspaces

$$V_1 = \ell^1(\mathbb{N}) \times \{0\}$$
$$V_2 = \{(x, y) \in V : ny_n = x_n \text{ for all } n \in \mathbb{N}\}.$$

Show that V_1 and V_2 are closed and that $V_1 + V_2 = \{v_1 + v_2 : v_1 \in V_1, v_2 \in V_2\}$ is not closed.

6. (**Norm with given unit ball**). Let $B \subset \mathbb{C}^d$ be a non-empty, open, convex, bounded subset with the property that $\alpha B = B$ for any $\alpha \in \mathbb{C}$ of absolute value one. Show that there exists a norm on $\mathbb{C}^d$ with (open) unit ball B.