Exercise sheet 2

1. (**Products and subspaces of Banach spaces**).

a) Let $(V, \|\cdot\|)$ be a Banach space and let W be a subspace. Show that $(W, \|\cdot\|)$ is a Banach space if and only if W is closed.

b) Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed vector spaces and equip the product space $V \times W$ with the norm

$$\|(v, w)\|_{V\times W} = \max\{\|v\|_V, \|w\|_W\}$$

for $(v, w) \in V \times W$ (see Exercise 1, Sheet 1). Then $(V \times W, \|\cdot\|_{V\times W})$ is a Banach space if and only if $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ are Banach spaces.

2. (**Quotient topology**). Let $(V, \|\cdot\|)$ be a normed vector space and let W be a closed subspace. Recall that the quotient topology for V/W (or more precisely the final topology for the quotient map $\pi : V \mapsto V/W,\ v \mapsto v + W$) is the finest (i.e. largest) topology on V/W for which π is continuous. Show that the quotient topology is the topology induced by the quotient norm $\|\cdot\|_{V/W}$ (cf. Lemma 2.15).

3. (**Achieving the quotient norm**). In this exercise we would like to show that the infimum in the definition of the quotient norm need not be achieved. For this, consider $V = C([-1, 1])$ with the norm $\|\cdot\|_\infty$ and the subspace

$$W = \left\{ f \in C([-1, 1]) : \int_{-1}^{0} f(x)\,\mathrm{d}x = \int_{0}^{1} f(x)\,\mathrm{d}x = 0 \right\}$$

a) Show that W is a closed subspace.

b) Let $f : x \in [-1, 1] \mapsto x$. Show that $\|f + W\|_{V/W} = \frac{1}{2}$.

c) Show that the infimum for $\|f+W\|_{V/W}$ is not achieved, i.e. there is no continuous function $g \in W$ such that $\|f + g\|_\infty = \frac{1}{2}$.

Turn the page.

4. (**Completion of a metric space**). Let (X, d) be a metric space. A completion of a metric space (X, d) is a pair consisting of a complete metric space (X^*, d^*) and an isometry $\iota : X \to X^*$ with dense image. Show that such a completion exists.

HINT: One approach is to directly generalize the proof of Theorem 2.32. Alternatively, one can consider for some fixed $x_0 \in X$ the map $x \in X \mapsto f_x \in B(X)$ where $B(X)$ is the Banach space of bounded, real-valued functions on X and $f_x : X \to \mathbb{R}$ is defined by $f_x(y) = \mathrm{d}(x, y) - \mathrm{d}(x_0, y)$ for $y \in X$.

5. (**More on ℓ^p-spaces**). For $1 \leq p < \infty$ we define the normed vector spaces $(\ell^p(\mathbb{N}), \|\cdot\|_p)$ in Sheet 1. For $p = \infty$ we let

$$\ell^\infty(\mathbb{N}) = \{x = (x_n)_n \in \mathbb{R}^{\mathbb{N}} : (x_n)_n \text{ is bounded}\}$$

be the vector space of bounded sequences in $\mathbb{R}$ and equip it with the norm given by $\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|$ for $x \in \ell^\infty(\mathbb{N})$.

a) Read the proof of Example 2.24(7) and apply it to show that $(\ell^p(\mathbb{N}), \|\cdot\|_p)$ is a Banach space for any $1 \leq p \leq \infty$.

b) Show that the closure in $\ell^\infty(\mathbb{N})$ of the subspace

$$c_c(\mathbb{N}) = \{x \in \mathbb{R}^{\mathbb{N}} : x_n = 0 \text{ for all but finitely many } n\}$$

is the subspace $c_0(\mathbb{N}) = \{x \in \mathbb{R}^{\mathbb{N}} : \lim_{n \to \infty} x_n = 0\}$. Note in comparison that for any $1 \leq p < \infty$ the subspace $c_c(\mathbb{N})$ is dense in $\ell^p(\mathbb{N})$.

c) Given p_1, p_2 with $1 \leq p_1 < p_2 \leq \infty$ prove the inequality $\|x\|_{p_1} \geq \|x\|_{p_2}$ for all $x \in \ell^{p_1}(\mathbb{N})$.

HINT: You can assume without loss generality that $\|x\|_{p_1} = 1$.

d) Show that $\ell^{p_1} \subsetneq \ell^{p_2}$ whenever $p_1, p_2 \in [1, \infty]$ satisfy $p_1 < p_2$.

e) Show that the norms $\|\cdot\|_p$ restricted to $c_c(\mathbb{N})$ are pairwise inequivalent.

f) Let $x \in \ell^q(\mathbb{N})$ for some $1 \leq q < \infty$. Prove that the limit $\lim_{p \to \infty} \|x\|_p$ exists and is equal to $\|x\|_\infty$.

6. (**Non-affine isometry**). Find a non-linear isometry $\varphi : (\mathbb{R}, |\cdot|) \to (\mathbb{R}^2, \|\cdot\|_\infty)$ with $\varphi(0) = 0$. Is this a contradiction to the theorem of Mazur and Ulam (Theorem 2.20)?