Prof. Dr. Manfred Einsiedler
Andreas Wieser

Exercise sheet 3

1. (**Operator norm**). Let $(V, \|\cdot\|_V), (W, \|\cdot\|_W)$ be two normed vector spaces and let $L : V \to W$ be linear. Show that L is a bounded operator if and only if it is Lipschitz-continuous and that in this case, $\|L\|_{\mathrm{op}}$ is the smallest Lipschitz constant for L.

2. (**Some explicit norms**). Compute the operator norms of the following maps:

a) The inclusion $\varphi : C^1([0,1]) \to C([0,1])$ where $C^1([0,1])$ (resp. $C([0,1])$ is equipped with the norm $\|\cdot\|_{C^1([0,1])}$ from Exercise 3, Sheet 1 (resp. the norm $\|\cdot\|_\infty$).

b) The restriction φ_0 of φ to the subspace $\{f \in C^1([0,1]) : f(0) = 0\}$.

c) The inclusion $\psi : C([0,1]) \to L^1_m([0,1])$ where m is the Lebesgue measure on the interval $[0,1]$.

d) The composition $\psi \circ \varphi_0$ of the maps in b) and c).

3. (**Two shift operators**). Consider the Banach space $\ell^2(\mathbb{N})$ (cf. Sheets 1 and 2) of $\mathbb{C}$-valued 2-summable sequences and the shift operators

$$\begin{aligned} L_{\text{left}} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), \quad & x = (x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots) \\ L_{\text{right}} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), \quad & x = (x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, \ldots). \end{aligned}$$

a) Show that $\|L_{\text{left}}\|_{\mathrm{op}} = 1$ and that L_{left} is not an isometry. Compute the eigenvalues of L_{left}.

b) Compute the eigenvalues of L_{right}.

4. (**Inequivalent norms on polynomials**). Given a compact, infinite subset $K \subset \mathbb{R}$ we define a norm[1] $\|\cdot\|_K$ on the polynomial ring $\mathbb{R}[x]$ via $\|p\|_K = \|p|_K\|_{C(K)}$. Show that for any two distinct compact, infinite subsets $K, L \subset \mathbb{R}$ the norms $\|\cdot\|_K, \|\cdot\|_L$ are inequivalent.

[1] This is indeed a norm and not only a semi-norm as any non-trivial polynomial has only finitely many zeros.

Turn the page.

5. (**Compact subsets of ℓ^p**). Let $p \in [1, \infty)$ Show that a subset $A \subset \ell^p(\mathbb{N})$ is compact (with respect to the $\|\cdot\|_p$-norm induced topology) if and only if it is closed, bounded and has uniformly small tails i.e. for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $x \in A$

$$\sum_{n=N}^{\infty} |x_n|^p < \epsilon.$$

Can you find and prove a similar statement characterizing compact subsets of $c_0(\mathbb{N})$?

6. (**Arzela-Ascoli on locally compact spaces**). Let (X, d) be a locally compact metric space and let $C_0(X)$ be the set of continuous functions vanishing[2] at ∞. (If you wish, you may also just consider $X = \mathbb{R}$ with the standard metric.) The vector space $C_0(X)$ is a closed subset of $C_b(X)$ and thus a Banach space when equipped with the norm $\|\cdot\|_\infty$. Extend the theorem of Arzela-Ascoli to describe the compact subsets of $C_0(X)$.

[2]That is, $f \in C_0(X)$ if for any $\epsilon > 0$ there is a compact set $K \subset X$ such that $|f(x)| < \epsilon$ for all $x \notin K$.