D-MATH | Functional analysis I | Fall 2018

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Exercise sheet 4

1. (**Continuity of the inner product**). Let V be a vector space (over $\mathbb{R}$ or $\mathbb{C}$) and let $\langle \cdot, \cdot \rangle$ be an inner product on V. Show that $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ (or $\mathbb{C}$) is a continuous map with respect to the norm topology (where the norm is induced by the inner product).

2. (**Non-unique approximation**). In each of the following cases find a Banach space $(V, \|\cdot\|)$, a non-empty closed convex subset $K \subset V$ and a point $v_0 \in V \setminus K$ that satisfy the required properties.

a) There are *several* points $k \in K$ with $\|v_0 - k\| = \inf_{k' \in K} \|v_0 - k'\|$.

b) There is *no* point $k \in K$ with $\|v_0 - k\| = \inf_{k' \in K} \|v_0 - k'\|$.

3. (**Polarization-identity**). In this exercise we discuss to which extent an inner product is determined by the norm it induces.

a) Let $(V, \langle \cdot, \cdot \rangle)$ be a real inner product space. Prove that for all $v, w \in V$

$$\langle v, w \rangle = \tfrac{1}{4}(\|v + w\|^2 - \|v - w\|^2).$$

b) Let $(V, \langle \cdot, \cdot \rangle)$ be a complex inner product space. Prove that for all $v, w \in V$

$$\langle v, w \rangle = \tfrac{1}{4} \sum_{k=0}^{3} \mathrm{i}^k \|v + \mathrm{i}^k w\|^2.$$

c) Let $(V, \|\cdot\|)$ be a real normed vector space and assume that the norm satisfies the parallelogram identity (cf. page 74). Find an inner product on V which induces the norm $\|\cdot\|$.

4. (**Hardy-space**). Let $D \subset \mathbb{C}$ be the open unit disc and let $p \in [1, \infty)$. Let V be the vector space of functions $f \in C(\overline{D})$ whose restriction $f|_D$ is holomorphic. For any $r \in (0, 1)$ we parametrize with $\gamma_r : [0, 1] \to \mathbb{C},\ t \mapsto r\mathrm{e}^{2\pi \mathrm{i} t}$ the circle of radius r around zero. Define for any $f \in V$

$$\|f\|_{H^p(D)} = \sup_{r \in (0,1)} \left(\int_0^1 |f(\gamma_r(t))|^p \, \mathrm{d}t \right)^{\frac{1}{p}}.$$

Turn the page.

a) Explain briefly why $\|\cdot\|_{H^p(D)}$ is a (well-defined) norm on V. The completion $H^p(D)$ of V with respect to this norm is a *Hardy-space* on the unit disc.

b) Show that for any $z \in D$ the map $f \in V \mapsto f(z) \in \mathbb{C}$ is continuous (with respect to the norm $\|\cdot\|_{H^p(D)}$ on V. Moreover, verify that for any open set $O \subset D$ with $\overline{O} \subset D$ the map

$$f \in V \mapsto f|_{\overline{O}} \in C(\overline{O})$$

is continuous.

c) Conclude from b) that there exists an injective linear map $H^p(D)$ into the space of holomorphic functions on D.

HINT: You may either use the general Hölder inequality (Theorem B.15) or restrict to the special case $p \in \{1, 2\}$.

5. (**Clarkson's inequality and uniform convexity of** L^p). Let $(X, \mathcal{B}, \mu)$ be a measure space and let $p \in [2, \infty)$.

a) Show that

$$\left\|\frac{f+g}{2}\right\|^p_{L^p_\mu(X)} + \left\|\frac{f-g}{2}\right\|^p_{L^p_\mu(X)} \leq \frac{1}{2}\left(\|f\|^p_{L^p_\mu(X)} + \|g\|^p_{L^p_\mu(X)}\right)$$

for $f, g \in L^p_\mu(X)$.

HINT: Show first that $a^p + b^p \leq (a^2+b^2)^{\frac{p}{2}}$ for $a, b > 0$.

b) Prove that $L^p_\mu(X)$ is uniformly convex.

c) Explain why $L^1_\mu(X)$ and $L^\infty_\mu(X)$ are not uniformly convex in general.

REMARK: Note that the spaces $L^p_\mu(X)$ for $p \in (1, 2)$ are also uniformly convex, which requires a generalization of Clarkson's inequality. This uniform convexity can be used to show that in most cases an isometric isomorphism between $L^1_\mu(X)$ and $L^p_\mu(X)$ does not exist.

6. (**Convolution on** ℓ^1). Show that the Banach space

$$\ell^1(\mathbb{Z}) = \left\{x = (x_n)_{n\in\mathbb{Z}} : \|x\|_1 = \sum_{n\in\mathbb{Z}} |x_n| < \infty\right\}$$

forms a commutative unital Banach algebra when equipped with the convolution $(x, y) \mapsto x * y$ given by $(x * y)_k = \sum_{j\in\mathbb{Z}} x_j y_{k-j}$ for all $k \in \mathbb{Z}$.