Exercise sheet 5

1. (**Dual Hilbert space**). Use the Fréchet-Riesz Representation Theorem to show that if $\mathcal{H}$ is a Hilbert space, then $\mathcal{H}^*$ is also a Hilbert space, and exhibit a natural isometric isomorphism between $\mathcal{H}$ and $(\mathcal{H}^*)^*$.

2. (**Haar measure on the torus**). In this exercise we would like to construct a Haar measure on the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. For this, recall that one can identify functions $\mathbb{T}^n \to \mathbb{C}$ with $\mathbb{Z}^n$-invariant functions $F : \mathbb{R}^n \to \mathbb{C}$ on $\mathbb{R}^n$ (i.e. we require $F(x+m) = F(x)$ for all $m \in \mathbb{Z}^n$). Furthermore, f is continuous (measurable) if and only if F is continuous (measurable). We define a measure m on $\mathbb{T}^n$ by requiring that

$$\int_{\mathbb{T}^n} f \,\mathrm{d}m = \int_{[0,1]^n} F \,\mathrm{d}m_{\mathbb{R}^n}$$

where $m_{\mathbb{R}^n}$ is the Lebesgue measure on $\mathbb{R}^n$ and f, F are measurable and correspond to each other. Show that m is a Haar measure on $\mathbb{T}^n$.

3. (**Counterexample to Fréchet-Riesz representation theorem**). Consider the vector space $V = c_c(\mathbb{N})$ with the inner product given by the $\ell^2(\mathbb{N})$-inner product. Notice that V is not complete as $c_c(\mathbb{N}) \subset \ell^2(\mathbb{N})$ is dense. Define the subspace

$$W = \Big\{ x \in V : \sum_{n=1}^{\infty} \tfrac{x_n}{n} = 0 \Big\}.$$

Show that W is closed and that the orthogonal complement $W^\perp$ in V is trivial. Deduce that the bounded linear functional $L : x \in V \mapsto \sum_{n=1}^{\infty} \frac{x_n}{n}$ cannot be represented as $\langle \cdot, y \rangle$ for any $y \in V$.

4. (**Lax-Milgram lemma**). Let $\mathcal{H}$ be a Hilbert space and let $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ (resp. to $\mathbb{C}$) be bilinear (resp. sesquilinear). Suppose that there is a constant $M > 0$ with the property that $|B(x, y)| \leq M \|x\| \|y\|$ for all $x, y \in \mathcal{H}$.

Turn the page.

a) Show that there exists a unique bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ with

$$B(x, y) = \langle Tx, y \rangle$$

for all $x, y \in \mathcal{H}$.

b) Assume additionally that B is *coercive* i.e. that there is a constant $c > 0$ such that $|B(x, x)| \geq c\|x\|^2$ for all $x \in \mathcal{H}$. Prove that T is invertible in this case and that the inverse is also bounded with $\|T^{-1}\|_{\mathrm{op}} \leq \frac{1}{c}$.

5. (**Integration against measurable functions**). Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and let ψ be a measurable function on X with the property that for any $g \in \mathcal{L}^2_\mu(X)$ the function ψg is integrable. Show that $\psi \in \mathcal{L}^2_\mu(X)$.

HINT: You may assume that ψ is non-negative or prove that it suffices to consider this case. Show by contradiction that the linear functional $g \in L^2_\mu(X) \mapsto \int_X \psi g \, \mathrm{d}\mu$ is bounded.

6. (**Mean ergodic theorem**). Let $\mathcal{H}$ be a Hilbert space and let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator. The mean ergodic theorem then states that for any $v \in \mathcal{H}$

$$\frac{1}{N} \sum_{n=0}^{N-1} U^n v \to P_I v$$

holds as $N \to \infty$ where P_I is the orthogonal projection onto $I = \{v \in \mathcal{H} : Uv = v\}$ (which is a closed subspace). Prove this.

HINT: Show the theorem first for vectors in I and for vectors of the form $Uw - w$ for $w \in \mathcal{H}$. Prove then that $\{Uw - w : w \in \mathcal{H}\}$ is dense in $I^\perp$.