Exercise sheet 6

1. (**Wirtinger's inequality**). Let $f \in C^1(\mathbb{T})$ with $\int_{\mathbb{T}} f(t)\,\mathrm{d}t = 0$. Show that

$$\|f\|_{L^2} \leq \tfrac{1}{2\pi}\|f'\|_{L^2}$$

with equality if and only if f is of the form $f(x) = a\mathrm{e}^{2\pi\mathrm{i}x} + b\mathrm{e}^{-2\pi\mathrm{i}x}$ for $a, b \in \mathbb{C}$.

2. (**Characters of finite abelian groups**). For any compact abelian (CA) metrizable group G let $\widehat{G}$ be the group of characters (unitary dual) of G.

a) For two CA metrizable groups G, H show that $\widehat{G \times H} \cong \widehat{G} \times \widehat{H}$.

b) Show that the unitary dual of $\mathbb{Z}/N\mathbb{Z}$ for any $N \in \mathbb{N}$ is isomorpic to $\mathbb{Z}/N\mathbb{Z}$.

HINT: For $k + N\mathbb{Z}$ you can consider the character $\chi_k : x \in \mathbb{Z}/N\mathbb{Z} \mapsto \mathrm{e}^{2\pi\mathrm{i}kx/N}$.

c) Use (a) and (b) to show that $\widehat{G} \cong G$ for any finite abelian group G (with the discrete topology).

3. (**Existence of groups with prescribed set of characters**). Given any countable abelian group Γ find a compact abelian (metrizable) group whose group of characters is isomorphic to Γ.

HINT: Consider $\mathbb{T}^\Gamma$ and the subset of points z given by the equations $z_{\gamma_1+\gamma_2} = z_{\gamma_1} + z_{\gamma_2}$ for $\gamma_1, \gamma_2 \in \Gamma$. If necessary, you can have a look at Exercise 3.53 and its hint.

4. (p**-adic integers**). Let p be a prime. We define

$$\mathbb{Z}_p = \left\{ (a_m)_m \in \prod_{m \geq 1} (\mathbb{Z}/p^m\mathbb{Z}) : a_{m+1} \equiv a_m \mod p^m \text{ for all } m \right\}$$

which by Tychonoff's theorem is a compact set when equipped with the topology induced by the product topology. Note that $\mathbb{Z}_p$ is an abelian group with pointwise addition.

Turn the page.

a) Show that $\mathbb{Z}_p$ is a metrizable compact abelian group.

b) Characterize the Haar measure on $\mathbb{Z}_p$.

HINT: Consider preimages of points under the projection maps $\pi_k : \mathbb{Z}_p \to \mathbb{Z}/p^k\mathbb{Z}$ for $k \in \mathbb{N}$.

c) Find all characters on $\mathbb{Z}_p$ and show the theorem on completeness of characters in this case.

5. (**Equidistribution and Benford's law**). A sequence $(x_n)_n$ in a metric space X is said to *equidistribute* to a Borel measure μ on X if

$$\frac{1}{n}\sum_{k=0}^{n-1} f(x_k) \to \int_X f \,\mathrm{d}\mu$$

as $n \to \infty$ for any $\mathbb{C}$-valued $f \in C_c(X)$. The goal of this exercise is to give an example of an equidistributing sequence and an application (Benford's law).

a) Let X be the interval $[0,1]$ and let μ be the Lebesgue measure on X. Show that a sequence $(x_n)_n$ in X equidistributes to μ if and only if

$$\frac{1}{n}\sum_{k=0}^{n-1} \mathrm{e}^{2\pi \mathrm{i} m x_k} \to 0$$

as $n \to \infty$ for any $0 \neq m \in \mathbb{Z}$.

HINT: Use that the subset $\mathcal{A} \subset C(\mathbb{T})$ of finite trigonometric sums $\sum_\ell' c_\ell \mathrm{e}^{2\pi \mathrm{i}\ell x}$ is dense.

From a) one can deduce that for any irrational number α the sequence $x_n = \{n\alpha\} = n\alpha - \lfloor n\alpha \rfloor$ equidistributes (to the Lebesgue measure) – see Example 2.49.

b) (Benford's law[1]) For any $n \in \mathbb{N}$ we let ℓ_n be the leading digit of 2^n. Show that for any $k \in \{1,\ldots,9\}$

$$\tfrac{1}{N}\big|\{n \in \{1,\ldots,N\} : \ell_n = k\}\big| \to \log_{10}\left(\tfrac{k+1}{k}\right)$$

as $N \to \infty$.

NOTE: Throughout this exercise you may use the results proven in Section 2.3.3. However, make sure that you explain how these are proven and fill in the gaps (like the hint in a)).

6. (**Derivatives in coordinate directions**). Let $d \in \mathbb{N}$ and let $C^k(\mathbb{T}^d)$ be the space of k-fold continuously differentiable functions. If $k < \infty$ we equip it with the norm $\|\cdot\|_{C^k}$ given by $\|f\|_{C^k} = \max_{\alpha \in \mathbb{N}^d : \|\alpha\|_1 \leq k} \|\partial_\alpha f\|_\infty$.

[1] The original discovery goes back to Newcomb (1881) who realized that in books on logarithm tables the leading digit 1 appears more often than 2, 2 more often than 3 and so on.

a) Explain why $C^k(\mathbb{T}^d)$ is a Banach space for $k < \infty$ (see Example 2.24(6)) and that $f_n \to f$ in $C^k(\mathbb{T}^d)$ if and only if $\partial_\alpha f_n \to \partial_\alpha f$ in $\|\cdot\|_\infty$ for all $\alpha \in \mathbb{N}^d$ with $\|\alpha\|_1 \le k$.

We want to relate smoothness properties of functions to decay properties of the Fourier coefficients.

b) Let $f \in C^\infty(\mathbb{T}^d)$. Show that for all $n \in \mathbb{Z}^d$ and all $\ell \in \mathbb{N}$

$$|a_n(f)| \ll_\ell \frac{1}{1 + \|n\|_2^{2\ell}}.$$

Conversely, prove that any $f \in L^2(\mathbb{T}^d)$ with this property has a smooth representative.

c) Let $f \in \mathcal{L}^2(\mathbb{T}^d)$ be such that the partial derivatives $\partial_j^\ell f$ exist and are continuous for all $j \in \{1, \dots, d\}$ and for all $\ell \in \mathbb{N}$. Show that f is smooth.