Exercise sheet 7

1. (**A symmetry for convolution**). Let G be an abelian locally compact metric group and let π be a unitary representation on a Hilbert space $\mathcal{H}$. Show that for any $v, w \in \mathcal{H}$ and $f \in L^1(G)$

$$\langle v, f *_\pi w \rangle = \langle f^* *_\pi v, w \rangle$$

where $f^* \in L^1(G)$ is given by $f^*(g) = \overline{f(g^{-1})}$ for $g \in G$.

2. (**The completeness assumption in Banach-Steinhaus**). Consider the normed vector space $V = c_c(\mathbb{N}) \subset \ell^\infty(\mathbb{N})$ of finitely supported sequences equipped with the supremum norm and define bounded operators $T_n \in B(V)$ by

$$T_n(x) = (x_1, 2x_2, \ldots, nx_n, 0, \ldots).$$

Show that the family $\{T_n : n \in \mathbb{N}\}$ is pointwise bounded and that $\sup_{n\in\mathbb{N}} \|T_n\|_{\mathrm{op}}$ is infinite.

3. (**Direct sums of Hilbert spaces**). For any $n \in \mathbb{N}$ let $\mathcal{H}_n$ be a Hilbert space.

a) Show that

$$\bigoplus_{n\in\mathbb{N}} \mathcal{H}_n = \left\{ (v_n)_n : v_n \in \mathcal{H}_n \text{ for any } n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} \|v_n\|^2_{\mathcal{H}_n} < \infty. \right\}$$

is a Hilbert space when equipped with the inner product given by

$$\langle (v_n)_n, (w_n)_n \rangle = \sum_{n=1}^{\infty} \langle v_n, w_n \rangle_{\mathcal{H}_n}$$

b) Suppose that the Hilbert spaces $\mathcal{H}_n$ are subspaces of a Hilbert space $\mathcal{H}$ (so that the inner products are compatible) and that $\mathcal{H}_m \perp \mathcal{H}_n$ for all $m \neq n$. Show that the closed linear hull of all $\mathcal{H}_n$ for $n \in \mathbb{N}$ in $\mathcal{H}$ is isomorphic to $\bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$ as a Hilbert space (i.e. there is an isomorphism which preserves the inner product).

4. (**The dihedral group**). Consider the dihedral group D_3 which is the group of isometries of an equilateral triangle in the plane centered at zero. Note that D_3 is generated by a rotation r by $120°$-degrees and a reflection s which satisfy $r^3 = s^2 = 1$ and $srs^{-1} = r^{-1}$. We let H be the subgroup generated by r.

a) Describe the characters on D_3 and show that they do not separate points.

HINT: The subgroup H is normal.

b) Let π be a unitary representation of D_3 on a Hilbert space $\mathcal{H}$. Restricting π to H we may apply Theorem 3.80. Denote by χ the character $H \to \mathbb{S}^1,\ r^k \mapsto \mathrm{e}^{2\pi \mathrm{i}k/3}$. Show that

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_\chi \oplus \mathcal{H}_{\chi^2}.$$

Can you give a quick independent argument for this?

c) Prove that the unitary operator π_s maps $\mathcal{H}_1$ to itself, $\mathcal{H}_\chi$ to $\mathcal{H}_{\chi^2}$ and $\mathcal{H}_{\chi^2}$ to $\mathcal{H}_{\chi^1}$.

5. (**More on the Banach algebra $\ell^1(\mathbb{Z})$**). Find a continuous injective algebra homomorphism $\ell^1(\mathbb{Z}) \to C(\mathbb{T})$ where $C(\mathbb{T})$ is a Banach algebra when endowed with pointwise multiplication and where the Banach algebra $\ell^1(\mathbb{Z})$ was introduced in Sheet 4.

6. (**Radial weights and pointwise convergence**). Recall from the lecture (see also Corollary 3.89) that any function $f \in L^2(\mathbb{R}^2)$ can be written as $f = \sum_{n\in\mathbb{Z}} f_n$ where every $f_n = \overline{\chi_n} * f \in L^2(\mathbb{R}^2)$ has the property that

$$f_n(k_\vartheta v) = \mathrm{e}^{2\pi \mathrm{i} n\vartheta} f_n(v)$$

for almost all $v \in \mathbb{R}^2$ and all $\vartheta \in \mathbb{T}$. Here, $k_\vartheta = \left(\begin{smallmatrix} \cos(2\pi\vartheta) & -\sin(2\pi\vartheta) \\ \sin(2\pi\vartheta) & \cos(2\pi\vartheta) \end{smallmatrix}\right)$. In this exercise, we want to discuss pointwise convergence of $\sum_{n\in\mathbb{Z}} f_n$.

a) Show that $f_n \in C(\mathbb{R}^2)$ for all $n \in \mathbb{N}$ whenever $f \in C(\mathbb{R}^2)$.

b) Prove that the series $\sum_{n\in\mathbb{Z}} f_n$ converges uniformly on compacta to the function f whenever $f \in C^1(\mathbb{R}^2)$.