Exercise sheet 8

1. (**Hellinger-Toeplitz theorem**). Let $\mathcal{H}$ be a Hilbert space and let $A : \mathcal{H} \to \mathcal{H}$ be a linear operator. Assume that A is self-adjoint in the sense that

$$\langle Ax, y\rangle = \langle x, Ay\rangle$$

for all $x, y \in \mathcal{H}$. Show that A is bounded.

HINT: This is really an exercise of the same direct nature as many other first exercises – if you find the right theorem to use.

2. (**Hamel-bases of Banach spaces**). Let X be an infinite-dimensional Banach space and let $\mathcal{B}$ be a Hamel basis of X (i.e. a basis in the sense of linear algebra). Show that $\mathcal{B}$ is uncountable.

HINT: Assuming that $\mathcal{B} = \{x_1, x_2, \ldots\}$ is countable, consider the increasing sequence of subspaces V_n where for $n \in \mathbb{N}$ the subspace V_n is spanned by $x_1, \ldots, x_n$.

3. (**The quotient norm and the annihilator**). Let X be a normed vector space and let Y be a subspace of X.

a) Show that the annihilator $Y^\perp = \{x^* \in X^* : x^*|_Y \equiv 0\}$ of Y is a closed subspace.

b) Show that for any $x \in X$

$$\max_{x^* \in Y^\perp : \|x^*\| \leq 1} |x^*(x)| = \inf_{y \in Y} \|x - y\|.$$

HINT: Revisit the proof of Corollary 7.6.

c) Exhibit a natural isometric isomorphism between $Y^\perp$ and the dual space of X/Y whenever Y is a closed subspace of X.

4. (**Separability of the dual space**). Assume X to be a normed vector space over $\mathbb{R}$. Prove that if the dual space X^* is separable then X is separable as well.

HINT: Let $\{x_n^*\} \subset X^*$ be a countable dense subset and choose for each x_n^* a unit vector $x_n \in X$ such that $x_n^*(x_n) \geq \frac{\|x_n^*\|}{2}$. Now consider the subspace $Y = \overline{\mathrm{span}_{\mathbb{Q}}\{x_n\}}$.

Turn the page.

5. (**Dual space of** ℓ^p). Let $p \in [1, \infty]$ and let q be Hölder conjugate[1] to p.

a) For $p < \infty$ the Hölder inequality (see Exercise 4b), Sheet 1) shows the operator $\phi_p : \ell^q(\mathbb{N}) \to \ell^p(\mathbb{N})^*$ defined for $x \in \ell^q(\mathbb{N})$ and $y \in \ell^p(\mathbb{N})$ by

$$\phi_p(x)(y) = \sum_{k=1}^{\infty} x_k y_k$$

is a bounded operator of norm at most 1. Show that ϕ_p is an isometric isomorphism.

b) For $p = \infty$ one analogously defines a map $\phi_\infty : \ell^1(\mathbb{N}) \to c_0(\mathbb{N})^*$ (see Exercise 5, Sheet 2 for the definition of $c_0(\mathbb{N})$). Show that ϕ_∞ is an isometric isomorphism.

c) Deduce that $c_0(\mathbb{N})$ is not reflexive.

6. (**Uniqueness in the Hahn-Banach theorem**).

a) Prove that if the dual space X^* of a real normed vector space X is strictly convex[2], then the Hahn-Banach extension of a continuous functional on a subspace to all of X is unique.

b) Give an explicit example where uniqueness of the Hahn-Banach theorem fails.

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[1]That is, with $\frac{1}{p} + \frac{1}{q} = 1$.

[2]A normed vector space V is strictly convex if the line segment between any two points $v_0, v_1 \in V$ with $\|v_0\| = \|v_1\| = 1$ only touches the unit sphere at the end points (i.e. $\|(1-t)v_0 + tv_1\| < 1$ for all $t \in (0, 1)$).