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Exercise sheet 9

1. (**Reflexivity of the dual**). Let X be a Banach space. Show that X is reflexive if and only if X^* is reflexive. Conclude that $\ell^1(\mathbb{N})$ is not reflexive.

HINT: If X^* is reflexive, assume that there is a point ℓ in $X^{**} \setminus \iota(X)$ where $\iota : X \to X^{**}$ is the natural embedding. Then use a corollary of the Hahn-Banach theorem.

2. (**Non-reflexivity of $L^1([0,1])$**). Let m be the Lebesgue measure on the interval $[0,1]$. Recall that the dual pairing between $L^1_m([0,1])$ and $L^\infty_m([0,1])$ yields an isometric embedding $L^1_m([0,1]) \to L^\infty_m([0,1])^*$. Construct an element in $L^\infty_m([0,1])^*$ that is not in the image of this isometry.

HINT: Partition the interval into countably many subintervals and use the Banach-limit.

3. (**Amenable groups**). Recall that a discrete group G is *amenable* if and only if there exists a linear functional $M \in (\ell^\infty(G))^*$ of norm one with the following properties:

(i) $M(a) \geq 0$ whenever $a \in \ell^\infty(G)$ is real-valued with $a \geq 0$.

(ii) $M(a^g) = M(a)$ for all $a \in \ell^\infty(G)$ and $g \in G$ where $a^g(h) = a(g^{-1}h)$.

Show that the additive group $\mathbb{Z}^n$ for any $n \in \mathbb{N}$ is amenable.

HINT: Imitate the proof of Corollary 7.14 and average over boxes.

4. (**Closed subspaces of reflexive spaces**). Let X be a normed vector space and let Y be a closed subspace.

a) Show that there is a natural isometric isomorphism $Y^* \simeq X^*/Y^\perp$ where $Y^\perp$ denotes the annihilator of Y as in Exercise 3, Sheet 8.

b) Prove that Y is reflexive if X is reflexive.

c) Show that $\ell^\infty(\mathbb{N})$ is not reflexive and deduce that $L^\infty_\mu(X)$ is not reflexive whenever $(X, \mathcal{B}, \mu)$ is a σ-finite measure space with no atoms.

Turn the page.

5. (**On the σ-finiteness assumption**). In this exercise we discuss the necessity of the σ-finiteness assumption in Proposition 7.34. Let X be an uncountable set and let $\mathcal{B}$ be the σ-algebra of subsets $A \subset X$ for which either A or $X \backslash A$ is countable. Furthermore, let μ be the counting measure on X.

a) Show that for any function $f \in L^1_\mu(X)$ there are at most countably many points $x \in X$ with $f(x) \neq 0$.

HINT: Since there are no non-trivial null-sets for the measure μ, $L^1_\mu(X)$ does indeed consist of functions and not only equivalence classes of integrable functions. For any $n \in \mathbb{N}$ consider the set $\{x \in X : |f(x)| > \frac{1}{n}\}$.

b) Given any function $f : X \to \mathbb{C}$ show that f is measurable if and only if there is a countable subset $A \subset X$ with the property that $f|_{X \backslash A}$ is constant.

c) Let $B(X)$ be the Banach space of bounded functions on X. Show that the dual pairing

$$L^1_\mu(X) \times B(X) \to \mathbb{C}, \quad (f, g) \mapsto \sum_{x \in X} f(x)g(x)$$

induces an isometric isomorphism $B(X) \simeq L^1_\mu(X)^*$. Note that by b) $L^\infty_\mu(X)$ is much smaller that $B(X)$ and in particular the natural pairing does **not** identify $L^\infty_\mu(X)$ with $L^1_\mu(X)^*$.

6. (**Complements of finite-dimensional subspaces**). Let V be a finite dimensional subspace of a real normed vector space space X. Prove that there exists a closed subspace W of X with $X = V \oplus W$.