D-MATH Functional analysis I Fall 2018
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Exercise sheet 10

1. (**Weak topology on finite-dimensional spaces**). Let X be a finite-dimensional normed vector space. Show that the weak topology on X coincides with the norm topology on X.

2. (**Riesz representation theorem – the locally compact case**). Read the proof of the Riesz representation theorem in the locally compact case (Section 7.4.4) and explain the main ideas.

3. (**Weak convergence and continuous maps**). Let X and Y be normed vector spaces and let $T : X \to Y$ be an operator.

a) Let $(x_n)_n$ be a weakly convergent sequence in X. Show that $(x_n)_n$ is bounded.

HINT: Use a theorem from Chapter 4.

b) Show that T is bounded if and only if for any weakly convergent sequence $(x_n)_n$ in X with limit $x \in X$ we have weak convergence $Tx_n \to Tx$ in Y.

4. (**Weak convergence in $\ell^1(\mathbb{N})$**). In this exercise we show that weak convergence of sequences in $\ell^1(\mathbb{N})$ coincides with strong convergence of sequences. Despite the result of this exercise, the weak topology on $\ell^1(\mathbb{N})$ is strictly weaker than the norm topology. Suppose that weak convergence does not imply strong convergence.

a) Show there exists a sequence $(a^{(n)})_n$ in $\ell^1(\mathbb{N})$ such that $\|a^{(n)}\|_1 = 1$ for all n and such that $(a^{(n)})_n$ converges weakly to 0 as $n \to \infty$. Conclude that $a_k^{(n)} \to 0$ for all $k \in \mathbb{N}$.

b) Select recursively a subsequence $(a^{(n_j)})_j$ and a strictly increasing sequence $(K_j)_j$ of indices such that

$$\sum_{k=1}^{K_{j-1}} |a_k^{(n_j)}| \leq \frac{1}{5} \quad \text{and} \quad \sum_{i=K_j+1}^{\infty} |a_k^{(n_j)}| \leq \frac{1}{5}.$$

Turn the page.

c) Let $b \in \ell^\infty(\mathbb{N})$ be such that $|b_k| = 1$ for all k and such that $a_k^{n_j} b_k = |a_k^{(n_j)}|$ holds for every j and for every k within the window $K_{j-1} < k \leq K_j$. Show that

$$\left| \sum_{k=1}^{\infty} a_k^{(n_j)} b_k \right| \geq \frac{1}{5}$$

for every $j \in \mathbb{N}$ and deduce a contradiction.

5. (**Unique ergodicity**). Let X be a compact metric space and let $T : X \to X$ be a homeomorphism. A (Borel) measure μ on X is said to be T-*invariant* if[1] $T_*\mu = \mu$. Assume that there is only one T-invariant probability measure μ on X (in this case, we say that T is *uniquely ergodic*). Show that for any $x \in X$ the orbit of x equidistributes, that is,

$$\tfrac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \to \int_X f \, \mathrm{d}\mu$$

as $N \to \infty$ for every $f \in C(X)$.

6. (**Metrizability of the weak topology**). Let X be an infinite-dimensional normed vector space. In this exercise we prove in particular that the weak topology on X is not metrizable.

a) Show that any non-empty open set in the weak topology is unbounded.

b) Show that the weak closure of the unit sphere $S = \{x \in X : \|x\| = 1\}$ is the closed unit ball $\overline{B_1^X} = \{x \in X : \|x\| \leq 1\}$.

c) Prove that there no countable neighborhood basis of $0 \in X$ for the weak-topology and deduce that the weak topology on X is not metrizable.

HINT: Use the ideas of the proof of Lemma 8.13. By Exercise 2, Sheet 8 there is no countable Hamel-basis of X^*.

[1] The pushforward measure $T_*\mu$ is defined by $T_*\mu(B) = \mu(T^{-1}(B))$ for every measurable $B \subset X$.