D-MATH | Functional analysis I | Fall 2018

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Exercise sheet 11

1. (**Space of $\mathbb{C}$-valued functions**). Let A be a set and let $\mathcal{F}(A)$ be the vector space of functions $A \to \mathbb{C}$. We define for $a \in A$ the semi-norm $\|f\|_a = |f(a)|$ for $f \in \mathcal{F}(A)$. Show that $\mathcal{F}(A)$ is a locally convex vector space.

2. (**Continuous linear functionals**). Let X be a locally convex vector space and let $\|\cdot\|_\alpha$ for $\alpha \in A$ be the attached collection of semi-norms. Show that a linear functional ℓ on X is continuous if and only if there exist finitely many $\alpha_1, \ldots, \alpha_N \in A$ and a constant $L > 0$ so that

$$|\ell(x)| \leq L \max_{n=1,\ldots,N} \|x\|_{\alpha_n}$$

for all $x \in X$.

3. (**The space** $\mathrm{MF}([0,1])$). Recall that a topological vector space is a vector space X with a topology so that addition and scalar multiplication are continuous operations. Let $\mathrm{MF}([0,1])$ be the space of $\mathbb{R}$-valued measurable functions identified on null-sets (for the Lebesgue measure λ) that we equip with the topology given by the neighborhoods

$$U_\varepsilon(f_0) = \Big\{ f \in \mathrm{MF}([0,1]) : \lambda(\{x \in [0,1] : |f(x) - f_0(x)| > \varepsilon\}) < \epsilon \Big\}.$$

Show that $\mathrm{MF}([0,1])$ is a Hausdorff topological vector space.

4. (**On local convexity**). In this exercise we would like to explain where the name *locally convex vector space* comes from. Let X be a locally convex vector space. A convex set $C \subset X$ is called *balanced* if $\lambda x \in C$ for any point $x \in C$ and scalar λ with $|\lambda| \leq 1$, and *absorbent* if for any $x \in X$ there is $\lambda > 0$ with $\lambda x \in C$.

a) Show that $0 \in X$ has a neighborhood basis of balanced absorbent convex sets.

b) Show that X is a topological vector space.

Turn the page.

5. (**Schwartz space**). Define the *Schwartz space* on $\mathbb{R}^d$ as

$$\mathscr{S}(\mathbb{R}^d) = \Big\{ f : \mathbb{R}^d \to \mathbb{C} : f \text{ is smooth and } \|x^\alpha \partial_\beta f\|_\infty < \infty \text{ for all } \alpha, \beta \in \mathbb{N}_0^d \Big\}.$$

Show that $\mathscr{S}(\mathbb{R}^d)$ is a Fréchet-space when equipped with the collection of semi-norms $f \mapsto \|x^\alpha \partial_\beta f\|_\infty$ for $\alpha, \beta \in \mathbb{N}_0^d$.

6. (**Convergence in** $C_c(U)$). Let $U \subset \mathbb{R}^d$ be open and consider the locally convex vector space $C_c(U)$ (see Example 8.63). Show that a sequence $(f_n)_n$ in $C_c(U)$ converges to $f \in C_c(U)$ if and only if there exists a compact set $K \subset U$ so that $\operatorname{supp}(f_n) \subset K$ for all $n \in \mathbb{N}$ and $\operatorname{supp}(f) \subset K$ and so that $f_n|_K \to f|_K$ uniformly on K.