Prof. Dr. Manfred Einsiedler
Andreas Wieser

Exercise sheet 13

1. (**Differentiation as a closed operator**). Show that the operator

$$\boldsymbol{\nabla} : H^1(\mathbb{T}^d) \to (L^2(\mathbb{T}^d))^d,\ f \mapsto (\boldsymbol{\partial}_1 f, \ldots, \boldsymbol{\partial}_d f)$$

is closed.

2. (**Existence of weak derivatives**). A function $f \in L^2(\mathbb{T}^d)$ is said to have an α-th (L^2-)*weak derivative* for $\alpha \in \mathbb{N}_0^d$ if there exists $g \in L^2(\mathbb{T}^d)$ such that

$$\int_{\mathbb{T}^d} \psi(x) g(x) \,\mathrm{d}x = (-1)^{\|\alpha\|_1} \int_{\mathbb{T}^d} (\partial_\alpha \psi)(x) f(x) \,\mathrm{d}x \tag{1}$$

holds for any test function $\psi \in C^\infty(\mathbb{T})$. If it exists, such a weak derivative is unique. Show that $f \in H^k(\mathbb{T})$ for some $k \in \mathbb{N}$ if and only if all α-th weak derivatives for $\alpha \in \mathbb{N}_0^d$ with $\|\alpha\|_1 \leq k$ exist.

3. (**Examples of weakly differentiable functions – I**). Let f be a continuous piecewise[1] C^1-function on the torus $\mathbb{T}$.

a) Show that the derivative f' (which is defined everywhere but at finitely many points) satisfies (1).

b) Show that $f \in H^1(\mathbb{T})$.

4. (**Examples of weakly differentiable functions – II**). Let $\varkappa > 0$ and $\delta \in (0, \frac{1}{2})$. Suppose that $f : \mathbb{T} \to \mathbb{R}$ is a function which satisfies $f(x) = x^\varkappa$ for $x \in [0, \delta]$, which is smooth on $(0, 1)$ and which is identically zero on $[1 - \delta, 1)$. Show that $f \in H^1(\mathbb{T})$ if and only if $\varkappa > \frac{1}{2}$.

[1] That is, a function with finitely many points where the derivative does not exists.

Turn the page.

5. (**Hölder continuous functions**). Let $f : \mathbb{T} \to \mathbb{C}$ be a function. We say that f is *Hölder continuous of exponent* $\beta \in \mathbb{R}_{\geq 0}$ if

$$|f(x) - f(y)| \leq |x - y|^\beta$$

where the distance $|x - y|$ is implicitly measured in the torus $\mathbb{T}$. In particular, Hölder continuous functions are continuous. The space of Hölder continuous functions of exponent β is denoted by $C^{0,\beta}(\mathbb{T})$ and can be turned into a Banach space by equipping it with the norm

$$\|f\|_{C^{0,\beta}} = \|f\|_\infty + \sup_{x,y \in \mathbb{T}} \frac{|f(x) - f(y)|}{|x - y|^\beta}.$$

The aim of this exercise is to show that any function $f \in H^1(\mathbb{T})$ has a representative which is Hölder continuous of exponent $\frac{1}{2}$.

a) (Morrey's inequality) Show that for any $f \in C^\infty(\mathbb{T})$ we have

$$|f(x) - f(y)| \leq \|f'\|_{L^2} |x - y|^{\frac{1}{2}}.$$

b) Apply a) to construct the canonical continuous embedding $H^1(\mathbb{T}) \to C^{0,\frac{1}{2}}(\mathbb{T})$.

c) Argue why any function $f \in H^1(\mathbb{T})$ has a representative which is Hölder continuous of exponent $\frac{1}{2}$.

HINT: For a) use the fundamental theorem of calculus. For b) and c) take a look at the proof of Theorem 5.6.

6. (**Examples and non-examples of compact operators**).

a) Show that the inclusion map $C(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ is not compact.

b) Let $U \subset \mathbb{R}^d$ be open. Show that the inclusion map $C_b^{k+1}(U) \to C_b^k(U)$ is compact if U is bounded. On the other hand, show that the inclusion is not compact if $U = \mathbb{R}$.

HINT: If U is bounded, apply the theorem of Arzela-Ascoli.

c) Show that the inclusion $H^1(\mathbb{T}) \to L^2(\mathbb{T})$ is compact.

7. (**Calkin algebra**). Let X be a Banach space.

a) Show that the set of compact operators $K(X) \subset B(X)$ is a two-sided ideal in $B(X)$.

HINT: This means that $K(X)$ is a linear subspace of $B(X)$ and that $LT, TL \in K(X)$ for all $L \in K(X)$ and $T \in B(X)$. The latter statement is the content of a lemma from the lecture.

b) Show that $B(X)/K(X)$ becomes a Banach algebra when equipped with the multiplication $(A + K(X))(B + K(X)) = AB + K(X)$ for all $A, B \in B(X)$ and with the quotient norm. This Banach algebra is called the *Calkin algebra* of X.

8. (**Compact operators and the strong operator topology**). In this exercise we would like to show that a limit of a sequence of compact operators in the strong operator topology does not need to be compact. For this, consider the Hilbert space $\ell^2(\mathbb{N})$. For $n \in \mathbb{N}$ let $e_n \in \ell^2(\mathbb{N})$ be the sequence with $(e_n)_k = \delta_{nk}$ for all $k \in \mathbb{N}$ and denote by P_n the projection onto the closed subspace spanned by $e_1, \ldots, e_n$.

a) Show that P_n is a compact operator for every $n \in \mathbb{N}$.

b) Prove that the sequence $(P_n)_n$ converges to the identity operator in the strong operator topology and argue why the latter operator is not compact.

9. (**An example of a Fredholm operator**). Let X be a Banach space and let $K : X \to X$ be a compact operator. Show that the image of the operator $\mathrm{id}_X - K$ is closed.

10. (**The adjoint operator**). Let $\mathcal{H}$ be a Hilbert space and $S, T : \mathcal{H} \to \mathcal{H}$ be bounded operators.

a) Show that $(aS + bT)^* = \bar{a}S^* + \bar{b}T^*$ and $(ST)^* = T^*S^*$ for all $a, b \in \mathbb{C}$.

b) Prove that $\mathrm{im}(T)^\perp = \ker(T^*)$ and $\ker(T)^\perp = \overline{\mathrm{im}(T^*)}$.

c) Show that T is unitary (that is, T is surjective and an isometry) if and only if $T^*T = TT^* = \mathrm{id}_{\mathcal{H}}$ is satisfied.

11. (**A spectral theorem for commuting operators**). Let $\mathcal{H}$ be a separable (infinite-dimensional) Hilbert space and let $T_1, T_2 : \mathcal{H} \to \mathcal{H}$ be two self-adjoint operators with $T_1T_2 = T_2T_1$ (that is, T_1 and T_2 *commute*). Assume that T_2 is compact with $\ker(T_2) = \{0\}$. State and prove a spectral theorem in this situation.

12. (**Green functions**). Consider the continuous integral kernel

$$(s,t) \in [0,1]^2 \mapsto G(s,t) = \begin{cases} s(t-1) & \text{if } s \leq t \\ t(s-1) & \text{if } t \leq s \end{cases}$$

as well as the induced compact self-adjoint integral operator

$$K : L^2([0,1]^2) \to C([0,1]^2) \subset L^2([0,1]^2).$$

Turn the page.

Compute all eigenvalues (and associated eigenfunctions) of K.

HINT: See Section 2.5.2.

NOTE: This exercise sheet will not be discussed in the exercise classes. However, the topics it treats are an important part of the course and in particular relevant for the exam.