D-MATH Functional analysis I Fall 2018
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Solutions for exercise sheet 0

1. **a)** Let τ be the intersection of all topologies on X which contain the preimages $f_\iota^{-1}(U_\iota)$ for any $\iota \in \mathcal{I}$ and any open set $U_\iota \subset X_\iota$. Then τ is a topology on X and satisifes all required properties.

b) See Lemma A.17.

2. Notice first that $\mathrm{d}(\cdot, A)$ is a continuous function for any $A \subset X$. Indeed, by the triangle inequality we have that

$$\mathrm{d}(x, A) \leq \mathrm{d}(x, a) \leq \mathrm{d}(x, y) + \mathrm{d}(y, a)$$

for any $x, y \in X$ and $a \in A$. Thus, $\mathrm{d}(x, A) \leq \mathrm{d}(x, y) + \mathrm{d}(y, A)$ and by symmetry of the argument

$$|\mathrm{d}(x, A) - \mathrm{d}(y, A)| \leq \mathrm{d}(x, y)$$

for any $x, y \in X$. This shows that $\mathrm{d}(\cdot, A)$ is 1-Lipschitz and in particular continuous.

Now assume that $A \subset X$ is closed. Then $\mathrm{d}(x, A) = 0$ if and only if $x \in A$. Surely, if $x \in A$ then $0 \leq \mathrm{d}(x, A) \leq \mathrm{d}(x, x) = 0$. Conversely, if $\mathrm{d}(x, A) = 0$ we may choose for any $k \in \mathbb{N}$ an element $a_k \in A$ with $\mathrm{d}(x, a_k) < \frac{1}{k}$. In particular, $\mathrm{d}(x, a_k) \to 0$ as $k \to \infty$ i.e. $(a_k)_k$ converges to x. As A is closed, $x \in A$.

Now let f be defined as in the exercise for two given non-empty closed disjoint subsets $A, B \subset X$. The second of the above claims shows that f is well-defined as $\mathrm{d}(x, A) + \mathrm{d}(x, B) = 0$ implies that $x \in A \cap B$, which is empty. Furthermore, f is continuous as the functions $\mathrm{d}(\cdot, A)$, $\mathrm{d}(\cdot, B)$ are continuous. It remains to show that f has the properties in Tychonoff's Theorem. For $x \in A$ we have

$$f(x) = \frac{0}{0 + \mathrm{d}(x, B)} = 0$$

and for $x \in B$ we have

$$f(x) = \frac{\mathrm{d}(x, A)}{\mathrm{d}(x, A)} = 1.$$

3. Let $X_1, X_2, \dots$ be countable collection of metric spaces where we denote by d_i the metric on X_i. A metric d inducing the product topology on $X = \prod_{i\in\mathbb{N}} X_i$ is then for instance given by

$$\mathrm{d}(x, y) = \sum_{i=1}^{\infty} 2^{-i} \min\{\mathrm{d}_i(x_i, y_i), 1\}$$

for any $x, y \in X$ (see Exercise 1b)). We will show that X is sequentially compact.

For this, notice first that a sequence $(x^{(n)})_n$ in X converges to some $x \in X$ if and only if $x_i^{(n)} \to x_i$ as $n \to \infty$ for any i. This can be shown directly from the definition of the product topology.

Now let $(x^{(n)})_n$ be an arbitrary sequence in X. By compactness of X_1 we may choose a subsequence $(x^{(n)})_{n\in\mathcal{J}_1}$ for $\mathcal{J}_1 \subset \mathbb{N}$ infinite with $x_1^{(n)} \to y_1 \in X_1$ as $n \to \infty$ with $n \in \mathcal{J}_1$. Similarly, there is a subsequence $(x^{(n)})_{n\in\mathcal{J}_2}$ of $(x^{(n)})_{n\in\mathcal{J}_1}$ with the property that $x_2^{(n)} \to y_2 \in X_2$ as $n \to \infty$ with $n \in \mathcal{J}_2$. Proceeding like this inductively, we obtain nested, infinite subsets

$$\mathcal{J}_1 \supset \mathcal{J}_2 \supset \mathcal{J}_3 \supset \dots$$

and a point $y = (y_i)_i \in X$ with

$$x_i^{(n)} \to y_i \quad (n \to \infty,\ n \in \mathcal{J}_i).$$

We now choose a sequence $(n_j)_j$ of natural numbers with $n_j \in \mathcal{J}_j$ and $n_j \leq n_{j+1}$ for all $j \in \mathbb{N}$. By our choices the subsequence $(x^{(n_j)})_j$ converges to $y \in X$.

4. **a)** If $(x_n)_n$ converges to $x \in X$ then an arbitrary neighborhood U of x contains all but finitely x_n so $U \in \mathcal{F}_{\text{tail}}$. This shows that $\mathcal{U}_x \subset \mathcal{F}_{\text{tail}}$ as claimed.

Conversely, if $\mathcal{U}_x \subset \mathcal{F}_{\text{tail}}$ any neighbourhood U of x (by definition of $\mathcal{F}_{\text{tail}}$) contains all but finitely many x_n so $x_n \to x$ as $n \to \infty$.

b) Assume first that $\mathcal{F}$ is an ultrafilter on X and let $A \subset X$. Consider

$$\mathcal{F}' = \{F' \subset X : A \cap F \subset F' \text{ for some } F \in \mathcal{F}\}.$$

If $A \cap F$ is non-empty for all $F \in \mathcal{F}$ the above defines a filter $\mathcal{F}'$ containing $\mathcal{F}$ and $\{A\}$. Since $\mathcal{F}$ is maximal, $\mathcal{F}' = \mathcal{F}$ and $A \in \mathcal{F}$. Similarly, if $(X \setminus A) \cap F$ is non-empty for all $F \in \mathcal{F}$ we have $X \setminus A \in \mathcal{F}$. Assuming that there are $F_1, F_2 \in \mathcal{F}$ such that the intersections $A \cap F_1$ and $(X \setminus A) \cap F_2$ are empty we obtain that $F_1 \subset X \setminus A$ and $F_2 \subset A$ so $\emptyset = F_1 \cap F_2 \in \mathcal{F}$ which is a contradiction.

Conversely, let $\mathcal{F}$ be a filter with $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$ for all $A \subset X$ and assume that $\mathcal{F}' \supset \mathcal{F}$ is a filter which strictly contains $\mathcal{F}$. Then there is $A \in \mathcal{F}'$

with $A \notin \mathcal{F}$. Therefore, $X \setminus A \in \mathcal{F}$ but since $\mathcal{F}' \supset \mathcal{F}$ we have $X \setminus A \in \mathcal{F}'$ so $A \cap (X \setminus A) = \emptyset \in \mathcal{F}'$ which is a contradiction. Thus, $\mathcal{F}$ is maximal.

Assume that $U \cap F \neq \emptyset$ for all $F \in \mathcal{F}$ and all neighborhoods of $x \in X$ and that $\mathcal{F}$ is an ultrafilter. Then any neighborhood U of x is in $\mathcal{F}$ (as otherwise $X \setminus U = F \in \mathcal{F}$ which contradicts $U \cap F \neq \emptyset$) and thus $\mathcal{F}$ converges to x.

c) Assume first that $\mathcal{F}$ is a filter on X (not necessarily maximal). We show that $f(\mathcal{F})$ is a filter on Y. Indeed, if $\emptyset \in f(\mathcal{F})$ then $f(F)$ is empty for some $F \in \mathcal{F}$ which implies that F is empty and contradicts the fact that $\mathcal{F}$ is a filter. If $B_1, B_2 \in f(\mathcal{F})$ then there are $F_1, F_2 \in \mathcal{F}$ with $f(F_1) \subset B_1$ and $f(F_2) \subset B_2$. Thus, $f(F_1 \cap F_2) \subset B_1 \cap B_2$ and hence $B_1 \cap B_2 \in f(\mathcal{F})$. If $B \in f(\mathcal{F})$ and $B' \supset B$ then there is $F \in \mathcal{F}$ with $f(F) \subset B \subset B'$ and thus $B' \in f(\mathcal{F})$.

Assume that $\mathcal{F}$ is an ultrafilter and let $B \in Y$. Follwing part b) we show that $B \in f(\mathcal{F})$ or $Y \setminus B \in f(\mathcal{F})$ In fact, either $f^{-1}(B) \in \mathcal{F}$ or $f^{-1}(Y \setminus B) \in \mathcal{F}$ (see b)). The former implies that $B \in f(\mathcal{F})$ as $f(f^{-1}(B)) \subset B$ and the latter implies that $Y \setminus B \in f(\mathcal{F})$ as $f(f^{-1}(Y \setminus B)) \subset Y \setminus B$.

Suppose now that $\mathcal{F}$ converges to $x \in X$ i.e. $\mathcal{U}_x \subset \mathcal{F}$. Let $V \subset Y$ be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is a neighborhood of x by continuity and therefore $f^{-1}(V) \in \mathcal{F}$. This shows that $V \in f(\mathcal{F})$ as $f(f^{-1}(V)) \subset V$.

d) Let $\mathcal{F}_0$ be a filter on X We consider the set

$$\mathcal{G} = \{\mathcal{F} : \mathcal{F} \text{ is a filter with } \mathcal{F} \supset \mathcal{F}_0\}$$

which we equip with the partial order $\preceq$ given by $\mathcal{F}_1 \preceq \mathcal{F}_2$ if and only if $\mathcal{F}_1 \subset \mathcal{F}_2$.

We use Zorn's lemma (see e.g. p. 538) to find a maximal element of $\mathcal{G}$ which is then an ultrafilter containing $\mathcal{F}_0$. Clearly, $\mathcal{G}$ is non-empty as $\mathcal{F}_0 \in \mathcal{A}$. For a linearly ordered subset $L \subset \mathcal{G}$ one checks that $\bigcup_{\mathcal{F} \in L} \mathcal{F}$ is a filter. Zorn's lemma now implies that $\mathcal{G}$ contains a maximal element as desired.

5. **a)** Before beginning the proof let us recall that X is compact if and only if for any collection $\mathcal{A}$ of closed subsets of X which satisfies the finite intersection property (that is, $A_1, \ldots, A_n \neq \emptyset$ for any $A_1, \ldots, A_n \in \mathcal{A}$) the intersection $\bigcap_{A \in \mathcal{A}} A$ is non-empty.

Assume first that X is compact and let $\mathcal{F}$ be an ultrafilter on X. We may thus choose a point x in the intersection $\bigcap_{F \in \mathcal{F}} \overline{F}$. Given a neighborhood U of x we have that $U \cap F \neq \emptyset$ for any $F \in \mathcal{F}$ as $x \in \overline{F}$. By 4b) $U \in \mathcal{F}$ and $\mathcal{F}$ converges to x.

Now assume that any ultrafilter on X converges and define for a collection $\mathcal{A}$ of closed subsets of X which satisfies the finite intersection property a filter as in the hint to the exercise. By 4d) we may choose an ultrafilter $\mathcal{F}'$ containing $\mathcal{F}$. Note that $\mathcal{A} \subset \mathcal{F}'$ by the way we defined the filter $\mathcal{F}$ and $\mathcal{F}'$. By assumption there is some $x \in X$ with $\mathcal{U}_x \subset \mathcal{F}'$. Since $\emptyset \notin \mathcal{F}'$ the intersection of any neighborhood of x and any element of $\mathcal{A}$ is non-empty. In other words, $A \cap U$ for $A \in \mathcal{A}$ and U a neighborhood of x is non-empty which shows that $x \in \overline{A} = A$ as A is closed. Thus, $x \in \bigcap_{A \in \mathcal{A}} A$.

b) As in the statement of Tychonoff's Theorem let X_ι for $\iota \in \mathcal{I}$ be topological spaces and let $X = \prod_{\iota \in \mathcal{I}} X_\iota$. Denote by π_ι the projection $X \to X_\iota$ for $\iota \in \mathcal{I}$.

If X is compact, X_ι for any $\iota \in \mathcal{I}$ is also compact as the image of a compact space under a continuous map.

So assume that X_ι is compact for any $\iota \in \mathcal{I}$. Let $\mathcal{F}$ be an ultrafilter on X. By 5a) we aim to show that $\mathcal{F}$ converges. For this, notice first that for any ι the image filter $\pi_\iota(\mathcal{F})$ is an ultrafilter (see 4c)) and converges (again by 5a)). So for any ι let $x_\iota \in X_\iota$ be such that $\pi_\iota(\mathcal{F})$ converges to x_ι. We define $x = (x_\iota)_\iota \in X$ and claim that $\mathcal{F}$ converges to x. By definition of the product topology it suffices to show that $\pi_\iota^{-1}(U_\iota)$ for $\iota \in \mathcal{I}$ and a neighborhood $U_\iota \subset X_\iota$ is an element of $\mathcal{F}$. This follows from the definition of the image filter and $\mathcal{U}_{x_\iota} \subset \pi_\iota(\mathcal{F})$.