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Solutions for exercise sheet 1

1. This is a straight-forward verification.

To show strict positivity let $(v, w) \in V \times W$. Then

$$\|(v, w)\|_{V\times W} = \max\{\|v\|_V, \|w\|_W\} = 0$$

is equivalent to $\|v\|_V = 0$ and $\|w\|_W = 0$ i.e. to $v = 0$ and $w = 0$ by strict positivity of the norms $\|\cdot\|_V, \|\cdot\|_W$.

To show homogeneity let $(v, w) \in V \times W$ and let α be a scalar. Then

$$\begin{aligned}\|\alpha(v, w)\|_{V\times W} &= \|(\alpha v, \alpha w)\|_{V\times W} = \max\{\|\alpha v\|_V, \|\alpha w\|_W\}\\ &= \max\{|\alpha|\|v\|_V, |\alpha|\|w\|_W\} = |\alpha| \max\{\|v\|_V, \|w\|_W\}\end{aligned}$$

as desired where in the second to last equality we used the homogeneity of the norms $\|\cdot\|_V, \|\cdot\|_W$.

To show the triangle inequality let $(v_1, w_1), (v_2, w_2) \in V \times W$. Then the triangle inequality for the norms $\|\cdot\|_V, \|\cdot\|_W$ yields

$$\begin{aligned}\|v_1 + v_2\|_V \leq \|v_1\|_V + \|v_2\|_V \leq \|(v_1, w_1)\|_{V\times W} + \|(v_2, w_2)\|_{V\times W},\\ \|w_1 + w_2\|_W \leq \|w_1\|_W + \|w_2\|_W \leq \|(v_1, w_1)\|_{V\times W} + \|(v_2, w_2)\|_{V\times W}.\end{aligned}$$

Thus,

$$\begin{aligned}\|(v_1, w_1) + (v_2, w_2)\|_{V\times W} &= \|(v_1 + v_2, w_1 + w_2)\|_{V\times W}\\ &= \max\{\|v_1 + v_2\|_V, \|w_1 + w_2\|_W\}\\ &\leq \|(v_1, w_1)\|_{V\times W} + \|(v_2, w_2)\|_{V\times W}\end{aligned}$$

as desired.

2. Assume first that $\|\cdot\|, \|\cdot\|'$ are equivalent. Let $c \geq 1$ be such that

$$\tfrac{1}{c}\|v\|' \leq \|v\| \leq c\|v\|'$$

for all $v \in V$. Let $(v_n)_n$ be a sequence in V which converges to $v \in V$ with respect to the norm $\|\cdot\|$ (or more precisely the topology induced by it). Then

$$0 \leq \limsup_{n\to\infty} \|v - v_n\|' \leq c \limsup_{n\to\infty} \|v - v_n\| = 0$$

so the limit $\lim_{n\to\infty} \|v - v_n\|'$ exists and is equal to zero. Thus, the sequence $(v_n)_n$ converges to $v \in V$ with respect to the norm $\|\cdot\|'$ as well. Similarly, one shows that any sequence $(v_n)_n$ in V which converges to $v \in V$ with respect to the norm $\|\cdot\|'$ also converges to $v \in V$ with respect to the norm $\|\cdot\|$.

We now turn to the proof of the converse claim. Assume first by contradiction that there is no constant c_1 such that $\|v\| \leq c_1\|v\|'$ for all $v \in V$. For any $n \in \mathbb{N}$ we may thus choose some $v_n \in V$ with $\|v_n\| > n\|v_n\|'$. After replacing v_n by $\frac{1}{\|v_n\|} v_n$ we may assume that $\|v_n\| = 1$ for all $n \in \mathbb{N}$. Since $\|v_n\|' < \frac{1}{n}$ for all n the sequence $(v_n)_n$ converges to zero with respect to the norm $\|\cdot\|'$. But since we have $\|v_n\| = 1$ for all $n \in \mathbb{N}$ it does not converge to zero with respect to the norm $\|\cdot\|$ which is a contradiction. Thus, there is a constant $c_1 \geq 1$ such that $\|v\| \leq c_1\|v\|'$ for all $v \in V$. Similarly, one shows that there is a constant $c_2 \geq 1$ such that $\|v\|' \leq c_2\|v\|$ for all $v \in V$. The constant $c = \max\{c_1, c_2\}$ then satisfies

$$\tfrac{1}{c}\|v\|' \leq \|v\| \leq c\|v\|'$$

for all $v \in V$ so the norms $\|\cdot\|, \|\cdot\|'$ are equivalent.

3. **a)** By Exercise 2 it suffices to find functions $f_n \in C^1([0,1])$ such that the sequence $(f_n)_n$ converges to zero with respect to the norm $\|\cdot\|_\infty$ but does not converge with respect to the norm $\|\cdot\|_{C^1([0,1])}$. For $n \in \mathbb{N}$ define $f_n : [0,1] \to \mathbb{R}$ via

$$f_n(x) = \tfrac{1}{n+1} x^{n+1}$$

for $x \in [0,1]$. Then certainly $f_n \in C^1([0,1])$ and we have $f_n'(x) = x^n$ for all $x \in [0,1]$. Thus,

$$\|f_n\|_\infty = \tfrac{1}{n}, \quad \|f_n\|_{C^1([0,1])} = \|f_n'\|_\infty = 1$$

so the functions f_n satisfy all desired properties.

b) It is straight-forward to verify that $\|\cdot\|_0 : C^1([0,1]) \to \mathbb{R}_{\geq 0}$ is homogeneous and satisfies the triangle inequality (i.e. it is a semi-norm). To show strict positivity, we use the fundamental theorem of calculus which states that for any $f \in C^1([0,1])$ we have

$$f(x) = f(0) + \int_0^1 f'(x)\,\mathrm{d}x \tag{1}$$

for all $x \in [0,1]$. So $\|f\|_0 = |f(0)| + \|f'\|_\infty = 0$ we have $f(0) = 0$ and $f' = 0$ and therefore $f = 0$ by (1). Thus, $\|\cdot\|_0$ is a norm.

To show equivalence of $\|\cdot\|_{C^1([0,1])}$ and $\|\cdot\|_0$ we also use the fundamental theorem of calculus. For $f \in C^1([0,1])$ and any $x \in [0,1]$ we have

$$|f(x)| \leq |f(0)| + \int_0^1 |f'(x)|\,\mathrm{d}x \leq |f(0)| + \int_0^1 \|f'\|_\infty\,\mathrm{d}x = \|f\|_0.$$

This shows that $\|f\|_\infty \leq \|f\|_0$ and together with the inequality $\|f'\|_\infty \leq \|f\|_0$ we obtain

$$\|f\|_{C^1([0,1])} \leq \|f\|_0$$

for all $f \in C^1([0,1])$. For the converse inequality we estimate directly

$$\|f\|_0 = |f(0)| + \|f'\|_\infty \leq \|f\|_\infty + \|f'\|_\infty \leq 2\|f\|_{C^1([0,1])}.$$

4. **a)** We fix $b \geq 0$ and consider the smooth function

$$f : a \in [0,\infty) \mapsto ab - \frac{a^p}{p}.$$

Let $a_{\max} \in (1,\infty)$ be a critical value of f i.e. with $f'(a_{\max}) = 0$. Then

$$f'(a) = b - a_{\max}^{p-1} = 0$$

and plugging this into the definition of f

$$f(a_{\max}) = a_{\max}b - \frac{a_{\max}b}{p} = \tfrac{1}{q}a_{\max}b = \tfrac{1}{q}b^{1+\frac{1}{p-1}} = \tfrac{1}{q}b^q$$

as $1 + \frac{1}{p-1} = \frac{p}{p-1} = \frac{1}{1-\frac{1}{p}} = q$. Since $\lim_{a\to\infty} f(a) = -\infty$ and $f(0) = 0 < f(a_{\max})$ we conclude that the unique (global) maximum of f is attained at the point $a_{\max}$. In summary, we have obtained that

$$f(a) \leq f(a_{\max}) = \tfrac{1}{q}b^q$$

for all $a \geq 0$ which is Young's inequality.

b) Let $x \in \ell^(\mathbb{N})$ and let $y \in \ell^q(\mathbb{N})$. We may assume without loss of generality that $\|x\|_p = \|y\|_q = 1$ by replacing x with $\frac{x}{\|x\|_p}$ or y with $\frac{y}{\|y\|_q}$ if necessary. By Young's inequality from a) we have

$$|x_n||y_n| \leq \frac{|x_n|^p}{p} + \frac{|y_n|^q}{q}$$

for any $n \in \mathbb{N}$. For a fixed (large) $N \in \mathbb{N}$ we may sum over all n between 1 and N to obtain

$$\sum_{n=1}^{N} |x_n||y_n| \leq \sum_{n=1}^{N} \frac{|x_n|^p}{p} + \sum_{n=1}^{N} \frac{|y_n|^q}{q} \leq \sum_{n=1}^{\infty} \frac{|x_n|^p}{p} + \sum_{n=1}^{\infty} \frac{|y_n|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1 \tag{2}$$

The series $\sum_{n=1}^{\infty} |x_n||y_n|$ is thus convergent and its value is less or equal than 1 as was to show.

c) Everything apart from the triangle inequality follows directly from the definition of $\|\cdot\|_p$. Let $x, y \in \ell^p(\mathbb{N})$. To avoid issues of convergence with fix a large enough integer N and consider sums only in the window $[1, N]$. Given any two sequences z, w one can apply b) to the sequences x', y' of finite support defined by $x'_n = z_n$, $y'_n = w_n$ if $n \leq N$, $x'_n = y'_n = 0$ if $n > N$ to obtain

$$\sum_{n=1}^{N} |z_n||w_n| \leq \left(\sum_{n=1}^{N} |z_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{N} |w_n|^q\right)^{\frac{1}{q}}.$$

Now notice that

$$\begin{aligned}\sum_{n=1}^{N} |x_n + y_n|^p &= \sum_{n=1}^{N} |x_n + y_n||x_n + y_n|^{p-1} \\ &\leq \sum_{n=1}^{N} |x_n||x_n + y_n|^{p-1} + \sum_{n=1}^{N} |y_n||x_n + y_n|^{p-1}.\end{aligned}$$

We now estimate each term. By (2) applied to the sequences $z = x$ and w defined by $w_n = |x_n + y_n|^{p-1}$ for $n \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{n=1}^{N} |x_n||x_n + y_n|^{p-1} &\leq \left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{N} |x_n + y_n|^{q(p-1)}\right)^{\frac{1}{q}} \\ &\leq \|x\|_p \left(\sum_{n=1}^{N} |x_n + y_n|^p\right)^{\frac{1}{q}}\end{aligned}$$

as $p = q(p-1)$. Similarly, one estimates the second term and obtains

$$\sum_{n=1}^{N} |x_n + y_n|^p \leq \|x\|_p \left(\sum_{n=1}^{N} |x_n + y_n|^p\right)^{\frac{1}{q}} + \|y\|_p \left(\sum_{n=1}^{N} |x_n + y_n|^p\right)^{\frac{1}{q}}.$$

Division by $\left(\sum_{n=1}^{N}|x_n+y_n|^p\right)^{\frac{1}{q}}$ gives

$$\left(\sum_{n=1}^{N}|x_n+y_n|^p\right)^{\frac{1}{p}} \leq \|x\|_p + \|y\|_p$$

and the limit as $N \to \infty$ yields the triangle inequality.

5. We begin by showing that V_1, V_2 are closed. Essentially, this is true as V_1, V_2 are defined by equations. For V_1, notice that it is the preimage of $\{0\}$ under the map

$$\Phi_1 : V \to \ell^1(\mathbb{N}), \quad (x,y) \mapsto y.$$

By definition of the norm in Exercise 1, Φ_1 is 1-Lipschitz and in particular continuous. Thus, V_1 is closed.

The subspace V_2 is the intersection over all $k \in \mathbb{N}$ of the subspaces

$$V_2^k = \{(x,y) \in V : ky_k = x_k\}.$$

Every V_2^k is closed as it is the preimage of $\{0\}$ under the continuous linear map

$$\Phi_2^k : V \to \mathbb{C}, \quad (x,y) \mapsto ky_k - x_k.$$

Indeed, Φ_2^k is $(k+1)$-Lipschitz as for any $(x,y) \in V$

$$\begin{aligned}|ky_k - x_k| &\leq k|y_k| + |x_k| \leq k\|y\|_1 + \|x\|_1 \leq k\|(x,y)\|_V + \|(x,y)\|_V \\ &= (k+1)\|(x,y)\|_V.\end{aligned}$$

It remains to show that $V_1 + V_2$ is not closed. For this we consider the subspace $c_c(\mathbb{N})$ of $\ell^1(\mathbb{N})$ of finitely supported sequences i.e. sequences x with $x_n = 0$ for all large enough n. Notice that $c_c(\mathbb{N})$ is dense in $\ell^1(\mathbb{N})$. Indeed, given $x \in \ell^1(\mathbb{N})$ we may choose for any $\epsilon > 0$ some $N \in \mathbb{N}$ with $\sum_{n=N+1}^{\infty}|x_n| < \epsilon$. If we then set $x' \in c_c(\mathbb{N})$ to be the sequence with $x'_n = x_n$ if $n \leq N$ and $x'_n = 0$ if $n > N$ we obtain

$$\|x - x'\|_1 = \sum_{n=N+1}^{\infty}|x_n| < \epsilon$$

As $c_c(\mathbb{N})$ is dense in $\ell^1(\mathbb{N})$, $c_c(\mathbb{N}) \times c_c(\mathbb{N})$ is dense in V.

We claim that $V_1 + V_2$ contains $c_c(\mathbb{N}) \times c_c(\mathbb{N})$ so that $V_1 + V_2$ is also dense. Indeed, given $(x,y) \in c_c(\mathbb{N})$ we can write for every n

$$(x_n, y_n) = (x_n - ny_n, 0) + (ny_n, y_n).$$

The first term on the right defines an element of V_1 and the second an element of V_2 as x, y are finitely supported. This shows the claim.

However, $V_1 + V_2$ is not all of V as for instance the vector (x, y) given by $x_n = 0$ and $y_n = \frac{1}{n^2}$ is not contained in $V_1 + V_2$. In fact, suppose we can write

$$(x, y) = (0, y) = (x', 0) + (x'', y'')$$

where $(x', 0) \in V_1$ and $(x'', y'') \in V_2$. Then $y''_n = y_n = \frac{1}{n^2}$ and $x'_n = x''_n = n y''_n = \frac{1}{n}$. But then $x' \in \ell^1(\mathbb{N})$ which yields a contradiction.

Summing things up, $V_1 + V_2$ is dense and not all of V. In particular, $V_1 + V_2$ is not closed.

6. Notice that $0 \in B$ as for any $b \in B$ we have $-b \in B$ (by rotational invariance) and thus $\frac{b+(-b)}{2} \in B$ (by convexity). If $v \in \mathbb{C}^d$ and $\alpha > 0$ is such that $\alpha v \in B$ then $0 \in B$ implies that $\beta v \in B$ for any $\beta \in [-\alpha, \alpha]$.

We define $\|\cdot\| : \mathbb{C}^d \to \mathbb{R}_{\geq 0}$ through

$$\|v\| = \inf\left\{\lambda > 0 : \tfrac{1}{\lambda} v \in B\right\}$$

for all $v \in \mathbb{C}^d$. To motivate this (at first possibly random) definition let us show that the open unit ball for $\|\cdot\|$ is B (though we haven't shown that $\|\cdot\|$ is a norm yet). If $v \in B$ then (as B is open) there exists a $\delta > 0$ such that $\frac{1}{1-\delta} v \in B$ so $\|v\| \leq 1 - \delta < 1$. Conversely, if $v \in \mathbb{C}^d$ satisfies $\|v\| < 1$ then any $\lambda > \|v\|$ fulfills $\frac{1}{\lambda} v \in B$ (here we use convexity) and in particular $v = \frac{1}{1} v \in B$.

We now turn to showing that $\|\cdot\|$ is indeed a norm. By definition $\|\cdot\|$ is non-negative. If $v \in B$ satisfies $\|v\| = 0$ then $\frac{1}{\lambda} v \in B$ for all $\lambda > 0$ which is impossible as B is bounded. This shows strict positivity.

To show homogeneity, let $\alpha \in \mathbb{C}$ be non-zero (otherwise we are already done) and let $v \in \mathbb{C}^d$. Write $\alpha = |\alpha|\alpha'$ where $\alpha' \in \mathbb{C}$ has norm one. Note first of all that $\|\alpha' v\| = \|v\|$ as by rotational invariance $\lambda > 0$ satisfies $\frac{1}{\lambda} v \in B$ if and only if it satisfies $\frac{1}{\lambda}\alpha' v \in B$. We can thus assume that $\alpha = |\alpha| > 0$. If $\lambda > 0$ is such that $\frac{1}{\lambda} v \in B$, the number $\lambda' = |\alpha|\lambda$ fulfills $\frac{1}{\lambda'}|\alpha| v \in B$ and so $\|\alpha\| v \leq |\alpha| \|v\|$. The other inequality is obtained in the same fashion.

To show the triangle inequality we let $v_1, v_2 \in \mathbb{C}^d$ be arbitrary and pick $\lambda_1, \lambda_2 > 0$ such that $\frac{1}{\lambda_1} v_1, \frac{1}{\lambda_2} v_2 \in B$. We need to show that $\|v_1 + v_2\| \leq \lambda_1 + \lambda_2$. Observe that

$$\tfrac{1}{\lambda_1+\lambda_2}(v_1 + v_2) = \tfrac{\lambda_1}{\lambda_1+\lambda_2}\left(\tfrac{1}{\lambda_1} v_1\right) + \tfrac{\lambda_2}{\lambda_1+\lambda_2}\left(\tfrac{1}{\lambda_2} v_2\right).$$

By the choice of λ_1 and λ_2 the right hand side is a convex combination of elements of B and is therefore in B. This proves the triangle inequality and we conclude that $\|\cdot\|$ is a norm with open unit ball B as desired.