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Solutions for exercise sheet 2

1. **a)** Assume first that $(W, \|\cdot\|)$ is complete and let $(w_n)_n$ be a sequence in W which converges in V to a point $w \in V$. Then $(w_n)_n$ is in particular a Cauchy sequence and therefore has a limit in W. By uniqueness of limits in V, this limit is equal to w so $w \in W$.

Assume that W is closed and let $(w_n)_n$ is a Cauchy sequence in W. In particular, $(w_n)_n$ is a Cauchy sequence in V and since V is complete, it has a limit $w \in V$. As W is closed, we must have $w \in W$ so the Cauchy sequence converges in W.

b) Assume first that $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ are Banach spaces. Let $((v_n, w_n))_n$ be a Cauchy sequence in $V \times W$. For $\epsilon > 0$ there is $N \in \mathbb{N}$ such that

$$\|(v_n - v_m, w_n - w_m)\|_{V\times W} = \|(v_n, w_n) - (v_m, w_m)\|_{V\times W} < \epsilon$$

for all $m, n \geq N$. In particular,

$$\|v_n - v_m\|_V \leq \|(v_n, w_n) - (v_m, w_m)\|_{V\times W} < \epsilon$$
$$\|w_n - w_m\|_W \leq \|(v_n, w_n) - (v_m, w_m)\|_{V\times W} < \epsilon$$

which shows that $(v_n)_n$ is a Cauchy sequence in V and $(w_n)_n$ is a Cauchy sequence in W. Since $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ are assumed to be complete, there is $v \in V$ with $v_n \to v$ as $n \to \infty$ and there is $w \in W$ with $w_n \to w$ as $n \to \infty$.

Conversely, if $(V \times W, \|\cdot\|_{V\times W}$ is a Banach space and $(v_n)_n$ is a Cauchy sequence we can consider the sequence $(v_n, 0)$ in $V \times W$. This is a Cauchy sequence and thus converges. If (v, w) is the limit, then (apart from the fact that w needs to be zero) we have

$$\|v_n - v\|_W \leq \|(v_n - v, w)\|_{V\times W}$$

where the right hand side goes to zero. Hence, $(v_n)_n$ converges to v. Similarly, one shows that any Cauchy sequence in W converges.

2. We first note that by definiton of the quotient norm, the quotient map

$$\pi : V \to V/W, \quad v \mapsto v + W$$

is 1-Lipschitz and in particular continuous. In particular, the quotient topology τ_{quot} contains the topology induced by the norm τ_{norm}.

For the other inclusion, let $U \subset V/W$ such that $U' = \pi^{-1}(U)$ is open (in the norm topology). Let $v_0 + W \in U$ and choose $\epsilon > 0$ such that $B_\epsilon(v_0) \subset U'$. Note that then for any $w \in W$

$$B_\epsilon(v_0 + w) = B_\epsilon(v_0) + w \subset U'$$

as $U' - w = U'$. We claim that $B_\epsilon(v_0 + W) \subset U$. Indeed, if $\|(v+W)-(v_0+W)\|_{V/W} < \epsilon$ there is $w \in W$ with $\|v - (v_0 + w)\| < \epsilon$ i.e.

$$v \in B_\epsilon(v_0 + w) \subset U'$$

so $v + W \in U$. This concludes the proof.

3. **a)** That W is a subspace follows from the linearity of the intregral. To see that W is closed we just need to check that the maps

$$f \in C([-1,1]) \mapsto \int_{-1}^{0} f(x)\,\mathrm{d}x$$
$$f \in C([-1,1]) \mapsto \int_{0}^{1} f(x)\,\mathrm{d}x$$

are continuous. In fact, they are both 1-Lipschitz as for any $f, g \in C([-1,1])$

$$\left| \int_{-1}^{0} f(x) - g(x)\,\mathrm{d}x \right| \leq \int_{-1}^{0} |f(x) - g(x)|\,\mathrm{d}x \leq \|f - g\|_\infty$$

and similarly for the second map.

b) Define a bounded function $g : [-1,1] \mapsto \mathbb{R}$ through $g(x) = x + \frac{1}{2}$ for $x < 0$ and $g(x) = x - \frac{1}{2}$ for $x \geq 0$. Then $|f(x) - g(x)| = \frac{1}{2}$ for any $x \in [-1,1]$ but notice that g is not continuous.

We now approximate g by continuous functions. For $\epsilon > 0$ define

$$h_\epsilon(x) = \begin{cases} x + \frac{1}{2} + \epsilon & x < -\delta \\ mx & -\delta \leq x \leq \delta \\ x - \frac{1}{2} - \epsilon & x > \delta \end{cases}$$

where m is chosen such that h is continuous and δ is chosen such that $h \in W$. Explicitly, $m = 1 - \frac{\epsilon + \frac{1}{2}}{\delta}$ and a computation shows that

$$\int_{-1}^{0} h_\epsilon(x)\,\mathrm{d}x = 0$$

is equivalent to $\delta = \frac{2\epsilon}{\epsilon+\frac{1}{2}}$ (which is in $[0,1]$ for $\epsilon > 0$ small enough).

Now one can estimate the norm $\|g - h_\epsilon\|_\infty$ or proceed directly to obtain $\|f - h_\epsilon\|_\infty < \frac{1}{2}+\epsilon$. Indeed, for $x \in [-1,-\delta]\cup[\delta,1]$ the estimate $|f(x)-h_\epsilon(x)| \le \frac{1}{2}+\epsilon$ is clear from the definition of h_ϵ and for $x \in [-\delta,\delta]$ we have

$$|f(x) - h_\epsilon(x)| \le |1-m||x| \le |1-m|\delta = \epsilon + \frac{1}{2}.$$

This shows that $\|f\|_W < \frac{1}{2} + \epsilon$. For the lower bound notice that

$$\frac{1}{2} = \int_0^1 (x + h(x))\,\mathrm{d}x \le \|f + h\|_\infty \tag{1}$$

for any $h \in W$. This concludes part b).

c) The inequality in (1) implies that $x + h(x) = \frac{1}{2}$ for all $x \ge 0$ if $h \in W$ was a function archieving the infimum. Since the analogous statement holds true on the interval $[-1,0]$ the function h is actually the function $-g$ defined in b), thus discontinuous in 0.

4. ALTERNATIVE 1:

Let Y be the set of Cauchy-sequences in X and define for $y, y' \in Y$

$$\delta(y,y') = \lim_{n\to\infty} \mathrm{d}(y_n, y'_n)$$

where the limit exists as both y, y' are Cauchy. The function δ is a metric except for the fact that there are distinct elements $y, y' \in Y$ with $\delta(y,y') = 0$ (i.e. δ is a semi-metric). The latter relation defines an equivalence relation $\sim$ on Y and we let $X^* = Y/\sim$. Then

$$\mathrm{d}^*([y]_\sim, [y']_\sim) = \delta(y,y')$$

for $y, y' \in Y$ defines a metric d^* on X^*. An inclusion ι of X into X^* is given by

$$\iota(x) = [(x,x,x,\ldots)]_\sim$$

and one checks as in the proof of Theorem 2.32 that all required properties are satisfied.

ALTERNATIVE 2:

Consider the function defined in the hint

$$\Phi : x \in X \mapsto f_x \in B(X)$$

where $x_0 \in X$ is fixed, $B(X)$ is the Banach space of bounded, real-valued functions on X and $f_x : X \to \mathbb{R}$ is defined by $f_x(y) = \mathrm{d}(x, y) - \mathrm{d}(x_0, y)$ for $y \in X$.

We first claim that Φ is an isometry. In fact, for $x \in X$ we have

$$\|f_x\|_\infty = \sup_{y \in X} |\mathrm{d}(x, y) - \mathrm{d}(x_0, y)| \le \mathrm{d}(x, x_0)$$

by the triangle inequality and conversely $\mathrm{d}(x, x_0) = |f_x(x_0)| \le \|f_x\|_\infty$. Hence, Φ is an isometry and in particular, injective and continuous.

Let X^* be the closure of the image of Φ in $B(X)$, which we equip with the metric d^* coming from the metric on $B(X)$ (the latter is induced by $\|\cdot\|_\infty$). The argument of 1a) shows that (X^*, d^*) is complete.

5. **a)** We refer directly to Example 2.24(7) for $p \in [1, \infty)$ and to Example 2.24(2) for $p = \infty$. Note that $\ell^p(\mathbb{N})$ can be viewed as $L^p_\mu(X)$ for the measure space $(X, \mathcal{B}, \mu)$ where $X = \mathbb{N}$, $\mathcal{B} = \mathcal{P}(X)$ and μ is the counting measure.

b) We show two things: (i) $c_0(\mathbb{N})$ is closed and (ii) $c_c(\mathbb{N})$ is dense in $c_0(\mathbb{N})$ with respect to the topology of ℓ^∞.

For (i) let $(x^{(k)})_k$ be a sequence in $c_0(\mathbb{N})$ and assume that $x^{(k)} \to x \in \ell^\infty(\mathbb{N})$ as $k \to \infty$. For $\epsilon > 0$ let $k \in \mathbb{N}$ be such that

$$\|x^{(k)} - x\|_\infty < \epsilon$$

(as $x^{(k)} \to x$ in ℓ^∞) and let $N \in \mathbb{N}$ be such that $|x_n^{(k)}| < \epsilon$ for all $n \ge N$ (as $x^{(k)} \in c_0(\mathbb{N})$). Then for any $n \ge N$

$$|x_n| \le |x_n^{(k)} - x_n| + |x_n^{(k)}| \le \|x^{(k)} - x\|_\infty + \epsilon < 2\epsilon.$$

This shows that $\lim_{n\to\infty} x_n = 0$ and hence $c_0(\mathbb{N})$ is closed.

For (ii) we let $x \in c_0(\mathbb{N})$ and $\epsilon > 0$. Let N be large enough such that $|x_n| < \epsilon$ for all $n \ge N$. We define $y \in c_c(\mathbb{N})$ by $y_n = x_n$ for $n < N$ and $y_n = 0$ for $n \ge N$. Then for any $n \in \mathbb{N}$ we have $|y_n - x_n| = 0$ if $n < N$ and $|y_n - x_n| = |x_n| < \epsilon$ otherwise. Thus, $\|y - x\|_\infty < \epsilon$.

c) Let $x \in \ell^{p_1}(\mathbb{N})$. To prove the desired inequality we can assume that $\|x\|_{p_1} = 1$: otherwise we replace x by $\frac{x}{\|x\|_{p_1}} = x'$ which has norm one and which satisfies $\|x'\|_{p_2} \le 1$ if and only if $\|x\|_{p_1} \ge \|x\|_{p_2}$ holds.

Notice that $\|x\|_{p_1} = 1$ implies that $|x_n| \le \|x\|_{p_1} = 1$ for all n. In particular, if $p_2 < \infty$ we obtain

$$|x_n|^{p_2} \le |x_n|^{p_1}$$

and summing over n yields the desired inequality. Otherwise, $|x_n| \le 1$ for all n directly implies $\|x\|_\infty \le 1$.

d) If $\|x\|_{p_1} < \infty$ then $\|x\|_{p_2} < \infty$ by c) so $\ell^{p_1} \subset \ell^{p_2}$.

To see that $\ell^{p_1} \neq \ell^{p_2}$ for $p_2 < \infty$ consider the sequence $x = (x_n)_n$ defined by $x_n = \frac{1}{n^{\frac{1}{p}}}$ for all $n \in \mathbb{N}$ where $p = \frac{p_1+p_2}{2} \in (p_1, p_2)$. Recall that the series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ for $\alpha \in \mathbb{R}$ is (absolutely) convergent if and only if $\alpha > 1$. Then

$$\sum_{n=1}^{\infty} |x_n|^{p_2} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{p_2}{p}}}$$

is convergent ($\alpha = \frac{p_2}{p} > 1$) and

$$\sum_{n=1}^{\infty} |x_n|^{p_1} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{p_1}{p}}}$$

is divergent ($\alpha = \frac{p_1}{p} < 1$) as desired.

If $p_2 = \infty$ we can take the constant sequence $x = (x_n)_n$ with $x_n = 1$ for all n, which is clearly bounded and thus element of ℓ^∞. Since it is not a nullsequence, $x \notin \ell^{p_1}$.

e) Suppose that $\|\cdot\|_{p_1}$ and $\|\cdot\|_{p_2}$ are equivalent and note that by c) the inequality $\|\cdot\|_{p_2} \leq \|\cdot\|_{p_1}$ cannot yield a contradiction. Let $c \geq 1$ with

$$\|x\|_{p_1} \leq \|x\|_{p_2} \tag{2}$$

for all $x \in c_c(\mathbb{N})$. For any $N \in \mathbb{N}$ we define $x^{(N)} \in c_c(\mathbb{N})$ by $x_n = 1$ if $n \leq N$ and $x_n = 0$ if $n > N$. Therefore, for any p

$$\|x^{(N)}\|_p = \left(\sum_{n=1}^{N} 1 \right)^{\frac{1}{p}} = N^{\frac{1}{p}}.$$

Plugging this into (2) we obtain $N^{\frac{1}{p_1}} \leq cN^{\frac{1}{p_2}}$ and therefore

$$N^{\frac{p_2}{p_1}} \leq c^{p_2}$$

Since $p_2 > p_1$ this cannot be true for all N as $N^{\frac{p_2}{p_1}} \to \infty$ as $N \to \infty$.

f) We first claim that it suffices to consider $x \in c_c(\mathbb{N})$. For this, recall that $c_c(\mathbb{N}) \subset \ell^p(\mathbb{N})$ is dense for any p. So given $x \in \ell^q(\mathbb{N})$ as in the statement of the exercise and given $\epsilon > 0$ we let $x' \in c_c(\mathbb{N})$ so that $\|x - x'\|_q < \epsilon$. By c) this implies $\|x - x'\|_p < \epsilon$ for any $\infty \geq p \geq q$. Therefore, by the triangle inequality

$$\|x'\|_p - \epsilon \leq \|x'\|_p - \|x - x'\|_p \leq \|x\|_p \leq \|x'\|_p + \|x - x'\|_p \leq \|x'\|_p + \epsilon. \tag{3}$$

Since we assume that $\lim_{n\to\infty}\|x'\|_p = \|x'\|_\infty$ is already known, this shows that

$$\limsup_{p\to\infty}\big(\|x\|_p - \|x\|_\infty\big) \le \limsup_{p\to\infty}\big((\|x'\|_p+\epsilon)-(\|x'\|_\infty-\epsilon)\big) = 2\epsilon$$

Since ϵ was arbitrary we have proven $\limsup_{p\to\infty}\big(\|x\|_p - \|x\|_\infty\big) = 0$. On the other hand, by c) we have

$$0 \le \liminf_{p\to\infty}\big(\|x\|_p - \|x\|_\infty\big) \le \limsup_{p\to\infty}\big(\|x\|_p - \|x\|_\infty\big).$$

This concludes the proof of the claim and we may assume that $x \in c_c(\mathbb{N})$.

For $x \in c_c(\mathbb{N})$ and any $p \in [1,\infty)$ we note that

$$\|x\|_p = \left(\sum_{n=1}^{N}|x_n|^p\right)^{\frac{1}{p}} = \|x\|_\infty\left(\sum_{n=1}^{N}\Big(\frac{|x_n|}{\|x\|_\infty}\Big)^p\right)^{\frac{1}{p}}$$

when $N \in \mathbb{N}$ is such that $x_n = 0$ for every $n > N$. Since $\frac{|x_n|}{\|x\|_\infty} \le 1$ for all n and equal to 1 for some n (here we use $x \in c_c(\mathbb{N})$), we have

$$\|x\|_\infty \le \|x\|_p \le \|x\|_\infty N^{\frac{1}{p}}.$$

Since $\lim_{p\to\infty} N^{\frac{1}{p}} = 1$, this shows the claim in the exercise.

6. Consider for a function $\alpha : \mathbb{R}\to\mathbb{R}$ with $\alpha(0)=0$ the map

$$\varphi : \mathbb{R}\to\mathbb{R}^2, \quad t \mapsto (t,\alpha(t)).$$

We now choose α correctly so that φ is a non-linear isometry. If for any $s,t\in\mathbb{R}$

$$|\alpha(s)-\alpha(t)| \le |s-t| \tag{4}$$

is satisfied, φ is an isometry as

$$\|\varphi(s)-\varphi(t)\|_\infty = \max\{|s-t|, |\alpha(s)-\alpha(t)|\} = |s-t|$$

for all $s,t\in\mathbb{R}$. It remains to choose α as in (4) so that φ is non-linear. For instance, if α is differentiable and the derivative is bounded by 1, the mean value theorem implies that (4) is satisfied. An example of such a function is $\sin : \mathbb{R}\to\mathbb{R}$, which is also non-linear.

The given example does not contradict the Theorem of Mazur and Ulam as φ is clearly not surjective and as $\mathbb{R}^2$ with the norm $\|\cdot\|_\infty$ is not strictly subadditive. The latter follows from the calculation

$$\|2e_1\|_\infty = \|e_1-e_2\|_\infty + \|e_1+e_2\|_\infty$$

where e_1, e_2 form the standard basis of $\mathbb{R}^2$.