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Solutions for exercise sheet 3

1. If L is bounded, any $v_1, v_2 \in V$ are distinct we can write

$$\|L(v_1) - L(v_2)\|_W = \|L(v_1 - v_2)\|_W = \|v_1 - v_2\|_V \|L(v)\|_W$$

where $v = \frac{v_1 - v_2}{\|v_1 - v_2\|_V}$ has norm one. By definition of the operatornorm

$$\|L(v_1) - L(v_2)\|_W = \|v_1 - v_2\|_V \|L(v)\|_W \leq \|v_1 - v_2\|_V \|L\|_{\mathrm{op}}$$

so L is $\|L\|_{\mathrm{op}}$-Lipschitz.

Conversely, if L is K-Lipschitz we have

$$\|L(v)\|_W = \|L(v) - L(0)\|_W \leq K\|v\|_V \leq K$$

for any $v \in V$ with $\|v\|_V \leq 1$. Thus, L is bounded and $\|L\|_{\mathrm{op}} \leq K$.

This discussion shows that a bounded operator L satisfies $\|L\|_{\mathrm{op}} \leq K$ and so $\|L\|_{\mathrm{op}}$ is indeed the smallest Lipschitz-constant.

2. **a)** Since $\|\cdot\|_{C^1([0,1])} \geq \|\cdot\|_\infty$ the operator norm $\|\varphi\|_{\mathrm{op}}$ is certainly bounded by 1 (see also Exercise 1). As the constant function $x \in [0,1] \mapsto 1$ has norm 1 for the C^1-norm as well as the supremum norm, the operator norm is exactly 1.

b) a) implies that $\|\varphi_0\|_{\mathrm{op}} \leq 1$. The polynomial function $p : x \in [0,1] \mapsto x$ satisfies $p' = 1$ so $\|p\|_\infty = 1 = \|p\|_{C^1([0,1])}$. This shows $\|\varphi_0\|_{\mathrm{op}} = 1$.

c) If $f \in C([0,1])$ we have

$$\|f\|_1 = \int_0^1 |f(x)| \,\mathrm{d}x \leq \|f\|_\infty$$

so $\|\psi\|_{\mathrm{op}} \leq 1$. Furthermore, $\|1\|_1 = \|1\|_\infty = 1$ so equality holds.

d) Lemma 2.57 implies that $\|\psi \circ \psi_0\|_{\mathrm{op}} \leq 1$. However, we claim that in this case $\|\psi \circ \psi_0\|_{\mathrm{op}} = \frac{1}{2}$. To see this, let $f \in C^1([0,1])$ with $f(0) = 0$ and with $\|f\|_{C^1([0,1])} \leq 1$. Then

$$\begin{aligned}\int_0^1 |f(x)| \,\mathrm{d}x = \int_0^1 \Big| \int_0^x f'(t) \,\mathrm{d}t \,\mathrm{d}x \Big| &\leq \int_0^1 \int_0^x |f'(t)| \,\mathrm{d}t \,\mathrm{d}x \\ &\leq \int_0^1 \int_0^x 1 \,\mathrm{d}t \,\mathrm{d}x = \frac{1}{2}.\end{aligned}$$

This shows that $\|\psi \circ \psi_0\|_{\mathrm{op}} \leq \frac{1}{2}$. Since equality $\|f\|_1 = \frac{1}{2}\|f\|_{C^1([0,1])}$ holds for the polynomial $f = p$ from b), we conclude $\|\psi \circ \psi_0\|_{\mathrm{op}} = \frac{1}{2}$.

3. **a)** Certainly, $\|L_{\mathrm{left}}\|_{\mathrm{op}} \leq 1$ as for any $x \in \ell^2(\mathbb{N})$

$$\sum_{n=2}^{\infty} |x_n|^2 \leq \sum_{n=1}^{\infty} |x_n|^2.$$

Equality holds whenever $x_1 = 0$ so $\|L_{\mathrm{left}}\|_{\mathrm{op}} = 1$. L_{left} is not an isometry, as it is not injective.

To compute the eigenvalues of L_{left} (if there are any), we assume that $L_{\mathrm{left}}(x) = \lambda x$ for some non-zero $x \in \ell^2(\mathbb{N})$ and $\lambda \in \mathbb{C}$. Then

$$(\lambda x_1, \lambda x_2, \lambda x_3, \ldots) = (x_2, x_3, x_4, \ldots)$$

and so $x_{k+1} = \lambda x_k$ for all $k \in \mathbb{N}$. Induction shows that $x_k = \lambda^{k-1} x_1$ for all $k \in \mathbb{N}$. This shows that

$$x = (x_1, \lambda x_1, \lambda^2 x_1, \ldots) = x_1(1, \lambda, \lambda^2, \ldots)$$

Since $x \neq 0$ and thus $x_1 \neq 0$ we may assume that $x_1 = 1$. Note that the series $\sum_{k=1} |\lambda|^{2k}$ is convergent by assumption (as it is equal to $\|x\|_2^2$) and therefore $|\lambda| < 1$. The set of eigenvalues of L_{left} is hence the open unit ball in the complex plane.

b) As in a) we assume that $L_{\mathrm{right}}(x) = \lambda x$ for some non-zero $x \in \ell^2(\mathbb{N})$ and $\lambda \in \mathbb{C}$. Then $\lambda \neq 0$ as L_{right} is injective and

$$(\lambda x_1, \lambda x_2, \lambda x_3, \ldots) = (0, x_1, x_2, \ldots).$$

Thus, $x_1 = 0$ as $\lambda x_1 = 0$, $x_2 = 0$ as $\lambda x_2 = x_1 = 0$ and so forth shows that $x = 0$ which is a contradiction. The operator L_{right} has hence no eigenvalues.

4. We may assume without loss of generality that there is a point $x_0 \in K \setminus L$ (otherwise we exchange K and L everywhere). Let $R > 0$ be such that the interval $[-R, R]$ contains K and L. By Urysohn's lemma (see Sheet 0) there is a function $f \in C([-R, R])$ with $f|_L = 0$ and $f(x_0) = 1$. For $n \in \mathbb{N}$ the Stone-Weierstrass Theorem implies that there is a polynomial $p_n \in \mathbb{R}[x]$ with $\|p_n|_{[-R,R]} - f\|_\infty < \frac{1}{n}$. This polynomial p_n satisfies $p_n(x_0) > 1 - \frac{1}{n}$ and $|p_n(x)| < \frac{1}{n}$ for all $x \in L$. This shows that $\|p_n\|_L \leq \frac{1}{n}$ and $\|p_n\|_K \geq 1 - \frac{1}{n}$ and thus (see for instance Sheet 1, Exercise 2) the norms $\|\cdot\|_K, \|\cdot\|_L$ are not equivalent.

5. Let us first show that any compact set $A \subset \ell^p(\mathbb{N})$ has uniformly small tails. This proves one direction of the claim as any compact set is closed (in any Hausdorff space) and bounded. Fix $\epsilon > 0$ and consider for every N the set

$$U_N = \left\{ x \in \ell^p(\mathbb{N}) : \sum_{n=N}^{\infty} |x_n|^p < \epsilon \right\}.$$

The collection of these open sets (which is increasing in N) for varying N covers $\ell^p(\mathbb{N})$ and in particular A. Thus, there is $N \in \mathbb{N}$ with $U_N \supset A$ which is what we claimed.

Now let $A \subset \ell^p(\mathbb{N})$ be a closed, bounded subset which has uniformly small tails. It suffices to show that a sequence $(x^{(n)})_n$ has a subsequence, which is a Cauchy sequence. Indeed, $\ell^p(\mathbb{N})$ is complete (see Sheet 2) and the so chosen subsequence must converge. The limit is in A so A is sequentially compact and thus compact.

To this end, notice that as for any k the sequence $(x_k^{(n)})_n$ has a convergent subsequence (and in particular, a Cauchy subsequence) as it is bounded[1]. In fact, there are n_j such that $x_k^{(n_j)}$ is a Cauchy-sequence for every k. Indeed, one chooses iteratively subsets $\mathcal{J}_1 \supset \mathcal{J}_2 \supset \ldots$ such that $(x_k^{(n)})_{n \in \mathcal{J}_k}$ is convergent and then considers a sequence $n_1 < n_2 < n_3 < \ldots$ with $n_k \in \mathcal{J}_k$ for every k. This diagonal argument follows more conceptually from Tychonoff's Theorem (see Sheet 0 and the solutions thereof).

To simplify notation, we assume that $(x_k^{(n)})_n$ is convergent for every k. Now let $\epsilon > 0$ and choose $K \in \mathbb{N}$ such that $\sum_{k=K+1}^{\infty} |x_k|^p < \frac{\epsilon^p}{2}$ for every $x \in A$. This implies that

$$\sum_{k=K+1}^{\infty} |x_k^{(n)} - x_k^{(m)}|^p < \epsilon^p$$

for every $m, n \in \mathbb{N}$. Also, we may fix $N \in \mathbb{N}$ such that

$$|x_k^{(n)} - x_k^{(m)}| < \frac{\epsilon}{K}$$

for all $k \in \{1, \ldots, K\}$ and all $m, n \geq N$. Overall we obtain

$$\sum_{k=1}^{\infty} |x_k^{(n)} - x_k^{(m)}|^p = \sum_{k=1}^{K} |x_k^{(n)} - x_k^{(m)}|^p + \sum_{k=K+1}^{\infty} |x_k^{(n)} - x_k^{(m)}|^p < \tfrac{K}{K^p}\epsilon^p + \epsilon^p < 2\epsilon^p$$

or in other words $\|x^{(n)} - x^{(m)}\|_p < 2^{\frac{1}{p}}\epsilon$ for all $m, n \geq N$. This shows that $(x^{(n)})_n$ is a Cauchy-sequence and thus we conclude that A is compact.

For $c_0(\mathbb{N})$ the analogous criterion can be applied and the proof is also analogous. Note that one can also apply Exercise 6 for $X = \mathbb{N}$.

[1] If $M > 0$ is such that $\|x\|_p \leq M$ for all $x \in A$, then $|x_k| \leq M$ for all $k \in \mathbb{N}$ and $x \in A$.

6. We claim that the following extended version is true.

Theorem (Arzela-Ascoli) Let (X, d) be a locally-compact metric space and let $A \subset C_0(X)$. Then A is compact if and only if the following three conditions hold: (i) A is closed and bounded, (ii) A is equicontinuous and (iii) A has uniform decay i.e. for every $\epsilon > 0$ there is a compact set $K \subset X$ such that $|f(x)| < \epsilon$ for all $f \in A$ and $x \notin K$.

Before turning to the proof let us note that there is a slight subtlety in the definition of equicontinuity. Indeed, in the lecture this notion was only introduced in the context of a compact metric space where any continuous function is uniformly continuous. We will call A *equicontinuous at a point* $x \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\mathrm{d}(y, x) < \delta \implies |f(y) - f(x)| < \epsilon$$

for all $f \in A$. Furthermore, A is *equicontinuous* if it is *equicontinuous* at every point in X. One checks that this notion of equicontinuity coincides with the other notion for the compact case.

We now turn to the proof of the theorem and assume first that A is compact. Condition (i) then follows immediately as any compact subset of a metric space is closed and bounded. Condition (iii) follows by considering for $\epsilon > 0$ fixed and a varying compact subset $K \subset X$ the open set

$$U_K = \{f \in C_0(X) : |f(x)| < \epsilon \text{ for all } x \notin K\}.$$

We thus obtain an open cover of X (and in particular of A) by definition of vanishing at infinity. So let $K_1, \ldots, K_m \subset X$ be compact sets with $A \subset U_{K_1} \cup \ldots \cup U_{K_m}$ and set $K = K_1 \cup \ldots \cup K_m$. If $f \in A$ and $x \notin K$ then we can pick some $1 \leq j \leq m$ with $f \in U_{K_j}$ and thus $|f(x)| < \epsilon$ as $x \notin K_j \subset K$.

It remains to show that A is equicontinuous. For this, we introduce for a given compact subset $K \subset X$ the bounded operator[2]

$$\Phi_K : C_0(X) \to C(K), \quad f \mapsto f|_K.$$

Since A is compact, the image $\Phi_K(A)$ is compact. If we apply this to a compact neighborhood K_x of a point $x \in X$ we obtain from the already proven version of Arzela-Ascoli that the family $f|_{K_x}$ for $f \in A$ is equicontinuous, which implies equicontinuity of A at x.

Assume now that A satisfies conditions (i)-(iii) and let $(f_n)_n$ be a sequence in A. As $C_0(X)$ is complete (see Example 2.24) it suffices to find a Cauchy-subsequence of

[2]In fact, this operator is surjective by the Tietze extension theorem – Proposition A.29.

$(f_n)_n$. For any $K \subset X$ compact, we see from the definitions that $(f_n|_K)$ is bounded and equicontinuous and thus (cf. Arzela-Ascoli for compact metric spaces) has Cauchy-subsequence[3].

We now choose for any $m \in \mathbb{N}$ a compact set K_m so that every f_n is bounded by $\frac{1}{m}$ on the complement of K_m (in absolute value of course). By replacing K_2 by $K_1 \cup K_2$, K_3 by $K_1 \cup K_2 \cup K_3$ and so forth (which preserves the defining property of these compact sets) we may assume that $K_1 \subset K_2 \subset K_3 \subset \dots$ holds. Using a diagonal argument (as in Exercise 5) we find a subsequence $(f_{n_j})_j$ of $(f_n)_n$ with the property that $f_{n_j}|_{K_m}$ is a Cauchy-sequence for any m.

We claim that $(f_{n_j})_j$ is a Cauchy-sequence in $C_0(X)$. Let $\epsilon > 0$ and choose $m \in \mathbb{N}$ with $\frac{1}{m} < \epsilon$. Then for any j_1, j_2 and $x \notin K_m$

$$|f_{n_{j_1}}(x) - f_{n_{j_2}}(x)| \leq |f_{n_{j_1}}(x)| + |f_{n_{j_2}}(x)| < 2\epsilon.$$

Combining this with the fact that the f_{n_j}'s are a Cauchy-sequence when restricted to K_m we obtain that $\|f_{n_{j_1}}(x) - f_{n_{j_2}}\|_\infty < 2\epsilon$ for all j_1, j_2 large enough and thus the theorem.

[3]In fact, if X is σ-compact one can find a subsequence $(f_{n_j})_j$ of $(f_n)_n$ with the property that $f_{n_j}|_K$ is a Cauchy-sequence for any $K \subset X$ compact.