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Solutions for exercise sheet 4

1. Let $(v_k)_k$ and $(w_k)_k$ be sequences in V with $v_k \to v$ and $w_k \to w$ as $k \to \infty$. We need to show that $\langle v_k, w_k\rangle \to \langle v, w\rangle$ as $k \to \infty$. For this, note that

$$\begin{aligned}\big|\, \langle v_k, w_k\rangle - \langle v, w\rangle \,\big| &\leq \big|\, \langle v_k, w_k\rangle - \langle v, w_k\rangle \,\big| + \big|\, \langle v, w_k\rangle - \langle v, w\rangle \,\big| \\ &\leq \big|\, \langle v_k - v, w_k\rangle \,\big| + \big|\, \langle v, w_k - w\rangle \,\big| \\ &\leq \|v_k - v\|\|w_k\| + \|v\|\|w_k - w\|.\end{aligned}$$

Since $\|w_k\| \to \|w\|$ as $k \to \infty$ we have $\|w_k\| \leq \|w\| + 1$ for all large enough k and in particular

$$\big|\, \langle v_k, w_k\rangle - \langle v, w\rangle \,\big| \leq \|v_k - v\|(\|w\| + 1) + \|v\|\|w_k - w\|.$$

Let $\epsilon > 0$. If $K \in \mathbb{N}$ is such that $\|v_k - v\|, \|w_k - w\| < \epsilon$ for all $k \geq K$ and such that $\|w_k\| \leq \|w\| + 1$, we have

$$\big|\, \langle v_k, w_k\rangle - \langle v, w\rangle \,\big| \leq (\|w\| + 1 + \|v\|)\epsilon$$

concluding the proof.

2. a) Let us define $V = \mathbb{R}^2$ with the norm $\|\cdot\| = \|\cdot\|_\infty$. Furthermore, consider the square $K = [-1, 1] \times [1, 2]$ as well as the point $v_0 = 0$. The closed ball $\overline{B_r}$ of radius r around 0 is the square $[-r, r]^2$ which intersects K only if $r \geq 1$ and otherwise has positive distance from it. If $r = 1$, $K \cap \overline{B_r} = [-1, 1] \times \{1\}$ so the distance from K to $v_0 = 0$ is 1 and is exactly achieved by all points in $[-1, 1] \times \{1\}$.

b) We already had this, see Exercise 3, Sheet 2.

3. a) For $v, w \in V$ we expand

$$\begin{aligned}\|v + w\|^2 &= \langle v + w, v + w\rangle = \langle v + w, v\rangle + \langle v + w, w\rangle \\ &= \langle v, v\rangle + \langle w, v\rangle + \langle v, w\rangle + \langle w, w\rangle = \|v\|^2 + 2\,\langle v, w\rangle + \|w\|^2\end{aligned}$$

by properties of the scalar product and similarly

$$\|v - w\|^2 = \|v\|^2 - 2\,\langle v, w\rangle + \|w\|^2.$$

Therefore,

$$\|v+w\|^2 - \|v-w\|^2 = 4\langle v,w\rangle$$

which proves the claim.

b) As in a) one shows that for $k \in \{0,1,2,3\}$ and $v, w \in V$

$$\begin{aligned}\|v+\mathrm{i}^k w\|^2 &= \langle v,v\rangle + \left\langle \mathrm{i}^k w, v\right\rangle + \left\langle v, \mathrm{i}^k w\right\rangle + \left\langle \mathrm{i}^k w, \mathrm{i}^k w\right\rangle \\ &= \|v\|^2 + \mathrm{i}^k\langle w,v\rangle + \mathrm{i}^{-k}\langle v,w\rangle + \mathrm{i}^k\mathrm{i}^{-k}\langle w,w\rangle \\ &= \|v\|^2 + \mathrm{i}^k\langle w,v\rangle + \mathrm{i}^{-k}\langle v,w\rangle + \|w\|^2.\end{aligned}$$

Therefore,

$$\sum_{k=0}^{3} \mathrm{i}^k\|v+\mathrm{i}^k w\|^2 = \sum_{k=0}^{3}\mathrm{i}^k\|v\|^2 + \sum_{k=0}^{3}\mathrm{i}^{2k}\langle w,v\rangle + \sum_{k=0}^{3}\langle w,v\rangle + \sum_{k=0}^{3}\mathrm{i}^k\|w\|^2.$$

Now notice that the sum $\sum_{k=0}^{3}\mathrm{i}^k$ is the sum over the points $1, \mathrm{i}, -1, -\mathrm{i}$ hence zero. Also, $\sum_{k=0}^{3}\mathrm{i}^{2k} = 1-1+1-1 = 0$ and so

$$\sum_{k=0}^{3}\mathrm{i}^k\|v+\mathrm{i}^k w\|^2 = \sum_{k=0}^{3}\langle w,v\rangle = 4\langle w,v\rangle$$

as desired.

c) Motivated by a) we define for $v, w \in V$

$$\langle v,w\rangle = \tfrac{1}{4}\big(\|v+w\|^2 - \|v-w\|^2\big)$$

where the second equality is the parallelogram identity and verify all the properties required of an inner product (see Definition 3.1). Certainly, strict positivity and symmetry follow directly from the definition of the inner product. It thus remains to prove linearity. Let $v_1, v_2, w \in V$ and compute

$$\begin{aligned}\langle v_1,w\rangle + \langle v_2,w\rangle &= \tfrac{1}{4}(\|v_1+w\|^2 - \|v_1-w\|^2) + \tfrac{1}{4}(\|v_2+w\|^2 - \|v_2-w\|^2) \\ &= \tfrac{1}{4}(\|v_1+w\|^2 + \|v_2+w\|^2) - \tfrac{1}{4}(\|v_1-w\|^2 + \|v_2-w\|^2) \\ &= \tfrac{1}{8}\big(\|v_1+v_2+2w\|^2 + \|(v_1+w)-(v_2+w)\|^2\big) \\ &\quad - \tfrac{1}{8}\big(\|v_1+v_2-2w\|^2 + \|(v_1-w)-(v_2-w)\|^2\big) \\ &= \tfrac{1}{8}\big(\|v_1+v_2+2w\|^2 + \|v_1-v_2\|^2\big) \\ &\quad - \tfrac{1}{8}\big(\|v_1+v_2-2w\|^2 + \|v_1-v_2\|^2\big) \\ &= \tfrac{1}{8}\big(\|v_1+v_2+2w\|^2 - \|v_1+v_2-2w\|^2\big) \\ &= \tfrac{1}{2}\langle v_1+v_2, 2w\rangle.\end{aligned}$$

We refer to this as equation (*). Applying (*) to $v_2 = 0$ we obtain in particular

$$\langle v_1, w \rangle = \tfrac{1}{2} \langle v_1, 2w \rangle . \tag{1}$$

We now turn to proving linearity by first showing it for rational multiples. Let $v, w \in V$. Let $n \in \mathbb{N}$ Applying (*) to $v_1 = (n-1)v$ and $v_2 = v$ and (1) to $v_1 = nv$ gives

$$\langle (n-1)v, w \rangle + \langle v, w \rangle = \tfrac{1}{2} \langle nv, 2w \rangle = \langle nv, w \rangle .$$

From this we conclude by induction that

$$n \langle v, w \rangle = \langle nv, w \rangle .$$

This can generalized to $n \in \mathbb{Z}$ by noting that $\langle -v, w \rangle = - \langle v, w \rangle$ follows from (*) applied to $v_1 = v$ and $v_2 = -v$. Replacing v by $\frac{1}{n}v$ we obtain $\left\langle \frac{1}{n}v, w \right\rangle = \frac{1}{n} \langle v, w \rangle$ and the combination of both statements yield for any $r \in \mathbb{Q}$

$$\langle rv, w \rangle = r \langle v, w \rangle .$$

Given $\alpha \in \mathbb{R}$ arbitrary we let r_n be a sequence of rational numbers converging to α.

By the above we only need to show that $\langle r_n v, w \rangle \to \langle \alpha v, w \rangle$ as $n \to \infty$. For this, we apply (*) to $v_1 = \alpha v$ and $v_2 = -r_n v$ and obtain

$$\langle \alpha v, w \rangle - \langle r_n v, w \rangle = \tfrac{1}{2} \langle (\alpha - r_n)v, 2w \rangle = \langle (\alpha - r_n)v, w \rangle .$$

We claim that the latter goes to zero, which would follow immediately if Cauchy-Schwarz-inequality was available (see Exercise 1). Since the proof of Cauchy-Schwarz uses $\mathbb{R}$-lineariy we instead apply the definition of the inner product, from which the statement follows directly.

4. **a)** Stricty positivity follows from the fact that if $\|f\| = \|f\|_{H^p(D)} = 0$ then $f \circ \gamma_r = 0$ for all r as a continuous function which vanishes almost everywhere, vanishes everywhere. Homogeneity is direct. Applying Minkowski's inequality to the function $f \circ \gamma_r$ yields the claim.

b) For any $f \in V$ and $z_0 \in D$ the Cauchy-integration formula implies that

$$f(z_0) = \tfrac{1}{2\pi \mathrm{i}} \int_{\gamma_r} \tfrac{f(z)}{z - z_0} \, \mathrm{d}z \tag{2}$$

holds whenever $z_0 \in B_r(0) = B_r$.

To show continuity of the evaluation maps, we use (2) to estimate for $z_0 \in D$ and $r < 1$ with $z_0 \in B_r$

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(\gamma_r(t))|}{|\gamma_r(t)-z_0|} |\gamma_r'(t)| \, \mathrm{d}t = \frac{r}{2\pi \, \mathrm{d}(\partial B_r, z_0)} \int_0^{2\pi} |f(\gamma_r(t))| \, \mathrm{d}t$$

where $\mathrm{d}(\partial B_r, z_0)$ denotes the distance of z_0 to the circle of radius r around zero (which is positive if r is close to one). Using the Hölder inequality (inserting a 1)

$$\int_0^{2\pi} |f(\gamma_r(t))| \, \mathrm{d}t \leq \left(\int_0^{2\pi} |f(\gamma_r(t))|^p \, \mathrm{d}t \right)^{\frac{1}{p}} \cdot \left(\int_0^{2\pi} 1 \, \mathrm{d}t \right)^{\frac{1}{q}} \leq (2\pi)^{\frac{1}{q}} \|f\|$$

where q is the Hölder conjugate to p and where we write $\|f\|$ for the Hardy-norm to simplify notation. We have thus shown that

$$|f(z_0)| \leq \frac{r(2\pi)^{\frac{1}{q}}}{2\pi \, \mathrm{d}(\partial B_r, z_0)} \|f\| \leq \frac{(2\pi)^{\frac{1}{q}}}{2\pi \, \mathrm{d}(\partial B_r, z_0)} \|f\|$$

and in particular that the evaluation maps are continuous.

If $O \subset D$ satisfies $\overline{O} \subset D$, considering the continuous map $z \in \overline{O} \mapsto \mathrm{d}(z, \partial B_1)$ shows that there is a radius $1 > r > 0$ such that $\overline{O} \subset B_r$. Thus, using this radius r above yields

$$\|f|_{\overline{O}}\|_\infty \leq \frac{(2\pi)^{\frac{1}{q}}}{2\pi \, \mathrm{d}(\partial B_r, z_0)} \|f\|$$

concluding the second part of b).

c) By Proposition 2.59 we may extend any map $f \in V \mapsto f|_{\overline{O}}$ as above to all of $H^p(D)$. We shall denote this by

$$\mathrm{Res}_{\overline{O}} : H^p(D) \to C(\overline{O}).$$

If $f \in H^p(D)$ there is a sequence $(f_n)_n$ of functions in V with $f_n \to f$ in $H^p(D)$. In particular, boundedness implies $\mathrm{Res}_{\overline{O}}(f_n) \to \mathrm{Res}_{\overline{O}}(f)$. The Cauchy integration formula shows that a uniform limit of holomorphic functions such as this one is also holomorphic.

This shows that any $f \in H^p(D)$ yields a holomorphic function on D, namely the function $g_f : D \to \mathbb{C}$ defined by $g_f(z) = \mathrm{Res}_{\overline{O}}(f)(z)$ whenever $z \in O$ and $\overline{O} \subset D$. Here, one verifies that this is indeed well-defined. This follows from the fact that if $O_1 \subset O_2$ the restriction functions satisfy

$$\mathrm{Res}_{\overline{O_2}}(f)|_{O_1} = \mathrm{Res}_{\overline{O_1}}(f). \tag{3}$$

Indeed, both sides define bounded operators $H^p(D) \to C(\overline{O_1})$ and are equal on V so the statement is a consequence of uniqueness in Proposition 2.59.

Notice that we do not know whether or not the function g_f extends continuously to the boundary. However, if we let D_r denote the open disc of radius r and V_r the vector space of continuous functions on $\overline{D_r}$ which are holomorphic in the interior, we obtain a continuous map

$$\mathrm{Res}_{\mathrm{tot}} : f \in H^p(D) \mapsto (g_f|_{\overline{D_r}})_r = (\mathrm{Res}_{\overline{D_r}}(f))_r \in \prod_{r\in(0,1)} V_r.$$

The image of $\mathrm{Res}_{\mathrm{tot}}$ lies in the subspace

$$W = \left\{ (f_r)_r \in \prod_{r\in(0,1)} V_r : f_{r_2}|_{\overline{D_{r_1}}} = f_{r_1} \text{ for all } 0 < r_1 < r_2 < 1 \right\}$$

by Equation 3. Since the discussion from a) and b) also apply to V_r we obtain a norm $\|\cdot\|_{H^p(D_r)}$ on V_r and may denote by $H^p(D_r)$ the completion of V_r. We also notice that W can be identified with the space of holomorphic functions on D.

It remains to show that $\mathrm{Res}_{\mathrm{tot}} : H^p(D) \to W$ is injective. For this, we would like to prove that

$$\|f\| = \sup_{r\in(0,1)} \|\mathrm{Res}_{\overline{D_r}}(f)\|_{H^p(D_r)} \tag{4}$$

for all $f \in V$ and that this formula extends to the completion. So if $f \in V$ is a function with $\mathrm{Res}_{\mathrm{tot}}(f) = 0$ we must have $\|f\| = 0$ and we would be done.

Note first that if $f \in V$ then by definition of the norm and the restriction map

$$\|\mathrm{Res}_{\overline{D_r}}(f)\|^p_{H^p(D_r)} = \sup_{s\in(0,r)} \int_0^{2\pi} |f(\gamma_s(t))|^p \,\mathrm{d}t$$

holds for any $r \in (0,1)$ so that (4) is satisfied. In particular, $\|\mathrm{Res}_{\overline{D_r}}(f)\|_{H^p(D_r)} \leq \|f\|_{H^p(D)}$. If $f \in H^p(D)$ is arbitrary, we need to refine this argument. Note that the induced map

$$\mathrm{Res}_{\overline{D_r}} : H^p(D) \to V_r \subset H^p(D_r)$$

has norm at most 1 when restricted to V by the above and thus (see Proposition 2.59) it has norm at most 1. Thus, $\|\mathrm{Res}_{\overline{D_r}}(f)\|_{H^p(D_r)} \leq \|f\|_{H^p(D)}$. To prove (4), we choose $g \in V$ with $\|f-g\|_{H^p(D)} < \epsilon$ and $r \in (0,1)$ with $\|\mathrm{Res}_{\overline{D_r}}(g)\|_{H^p(D_r)} > \|g\|_{H^p(D)} - \epsilon$. Then

$$\begin{aligned}\|\mathrm{Res}_{\overline{D_r}}(f)\|_{H^p(D_r)} &\geq \|\mathrm{Res}_{\overline{D_r}}(g)\|_{H^p(D_r)} - \|\mathrm{Res}_{\overline{D_r}}(f-g)\|_{H^p(D_r)} \\ &\geq \|g\|_{H^p(D)} - \epsilon - \|f-g\|_{H^p(D)} \\ &\geq \|f\|_{H^p(D)} - 3\epsilon\end{aligned}$$

which proves (4).

5. a) We first prove the statement in the hint. For this, we consider for any fixed $a > 0$ the function

$$\psi_a : b \in [0,\infty) \mapsto (a^2 + b^2)^{\frac{p}{2}} - b^p \in [0,\infty)$$

with $\psi_a(0) = a^p$. The derivative satisfies

$$\psi_a'(b) = \frac{p}{2}(a^2 + b^2)^{\frac{p}{2}-1} 2b - p b^{p-1}$$

and is thus non-negative whenever $(a^2+b^2)^{\frac{p}{2}-1} \geq b^{p-2}$. Since the latter is always true, ψ_a is monotonely increasing. This proves the claim in the hint.

We now turn to the proof of Clarkson's inequality and let $f, g \in L^p_\mu(X)$. For μ-almost every x we may apply the claim to $a = \frac{|f(x)|+|g(x)|}{2}$ and $b = \frac{\left||f(x)|-|g(x)|\right|}{2}$ (here strictly speaking one should choose a representative of f, g) to get

$$\begin{aligned}\left(\frac{|f(x)| + |g(x)|}{2}\right)^p + \left(\frac{|f(x)| - |g(x)|}{2}\right)^p &\leq \left(\tfrac{1}{4}2|f(x)|^2 + \tfrac{1}{4}2|g(x)|^2\right)^{\frac{p}{2}} \\ &= \left(\tfrac{1}{2}|f(x)|^2 + \tfrac{1}{2}|g(x)|^2\right)^{\frac{p}{2}} \\ &\leq \tfrac{1}{2}|f(x)|^p + \tfrac{1}{2}|g(x)|^p\end{aligned}$$

by convexity of $t \mapsto t^{\frac{p}{2}}$. Integrating over this yields the statement in a).

b) The statement in a) shows that for any $f, g \in L^p_\mu(X)$ in the closed unit ball

$$\left\|\frac{f+g}{2}\right\|^p \leq \tfrac{1}{2}(\|f\|^p + \|g\|^p) - \left\|\frac{f-g}{2}\right\|^p \leq 1 - 2^{-p}\|f-g\|^p$$

so uniform convexity is satisfied with the function

$$\eta(x) = 1 - \left(1 - 2^{-p}x^p\right)^{\frac{1}{p}}$$

as one readily checks.

c) Assume that there are two disjoint measurable sets $A, B \in \mathcal{B}$ with positive finite measure. Note that this is quite a weak assumption!

We denote by χ_A and χ_B the respective characteristic functions.

To see that $L^1 = L^1_\mu(X)$ is not uniformly continuous, one may consider

$$f = \tfrac{1}{\mu(A)}, \quad g = \tfrac{1}{\mu(B)}\chi_B$$

which are functions of norm one with

$$\tfrac{1}{2}\|f+g\|_{L^1} = \tfrac{1}{2}(\|f\|_{L^1} + \|g\|_{L^1}) = 1$$

as A, B are disjoint.

To see that $L^\infty = L^\infty_m u(X)$ is not uniformly convex consider the L^∞-norm one functions

$$f = \chi_A + \chi_B, \quad g = \chi_A - \chi_B$$

for which

$$\frac{f+g}{2} = \chi_A$$

also has norm one.

6. We verify first that the inner product is indeed well-defined. For $x, y \in \ell^1(\mathbb{Z})$ we have

$$\|x * y\|_1 = \sum_{k\in\mathbb{Z}} \Big| \sum_{j\in\mathbb{Z}} x_j y_{k-j} \Big| \leq \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} |x_j||y_{k-j}|.$$

By (absolute) convergence of the series $\sum_n |x_n|$ and $\sum_n |y_n|$ we may write (this is a standard fact from analysis) the latter as

$$\sum_{j\in\mathbb{Z}} |x_j| \Big(\sum_{k\in\mathbb{Z}} |y_{k-j}| \Big) = \sum_{j\in\mathbb{Z}} |x_j| \Big(\sum_{k\in\mathbb{Z}} |y_k| \Big) = \|x\|_1 \|y\|_1$$

We have thus also shown the required inequality for Banach algebras. Note that one should also check the bilinearity of the operation, which is however straight-forward; we thus omit it here.

It remains to check that $\ell^1(\mathbb{Z})$ is indeed unital and commutative. Let us first show that it is indeed commutative. For $x, y \in \ell^1(\mathbb{Z})$ and $k \in \mathbb{Z}$ we have

$$(x * y)_k = \sum_{j\in\mathbb{Z}} x_j y_{k-j} \underset{j'=k-j}{=} \sum_{j'\in\mathbb{Z}} x_{k-j'} y_{j'} = (y * x)_k$$

where the substitution is justified by absolute convergence.

Define $\delta \in \ell^1(\mathbb{Z})$ through

$$\delta_n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{else} \end{cases}.$$

Then one computes for any $x \in \ell^1(\mathbb{Z})$ and $k \in \mathbb{Z}$

$$(x * \delta)_k = \sum_{j\in\mathbb{Z}} x_j \delta_{k-j} = x_k$$

so $x * \delta = x$ and $\delta * x = x$ follows from commutativity.