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Solutions for exercise sheet 5

1. Recall that the Frechet-Riesz theorem states that

$$\mathcal{H} \to \mathcal{H}^*, \quad x \mapsto \phi_x = \langle \cdot, x \rangle$$

defines a semi-linear isomorphism between $\mathcal{H}$ and $\mathcal{H}^*$. We denote by $\phi \mapsto x_\phi$ the semilinear inverse and define for $\phi, \phi' \in \mathcal{H}^*$

$$\langle \phi, \phi' \rangle_{\mathcal{H}^*} = \langle x_{\phi'}, x_\phi \rangle .$$

One checks that this defines an inner product on $\mathcal{H}^*$ which induces the operator norm. Since $x \mapsto \phi_x$ is isometric, $\mathcal{H}^*$ is also a Banach space with respect to the operator norm.

It remains to exhibit the natural isometric isomorphism between $\mathcal{H}$ and $\mathcal{H}^{**}$. For $x \in \mathcal{H}$ we can define

$$\psi_x : \phi \in \mathcal{H}^* \mapsto \phi(x)$$

to obtain a linear map

$$\Psi : \mathcal{H} \to \mathcal{H}^{**}, \quad x \mapsto \psi_x$$

We need to show that this is an isometric isomorphism. To see that it is isometric, let $x \in \mathcal{H}$ be non-zero (otherwise there is nothing to show) and note that

$$\|\psi_x\|_{\mathrm{op}} = \sup_{\|\phi\| \leq 1} |\psi_x(\phi)| = \sup_{\|\phi\| \leq 1} |\phi(x)| \leq \sup_{\|\phi\| \leq 1} \|\phi\| \|x\| = \|x\|$$

as well as

$$\|\psi_x\|_{\mathrm{op}} \geq \left| \tfrac{1}{\|x\|} \phi_x(x) \right| = \|x\|.$$

In particular, Ψ is injective. To see that it is surjective, we apply the first part of the exercise and the Frechet-Riesz theorem (twice). So let $\psi \in \mathcal{H}^{**}$ and write $\psi = \langle \cdot, \phi \rangle_{\mathcal{H}^*}$ for $\phi \in \mathcal{H}^*$. Further, write $\phi = \phi_x$ for $x \in \mathcal{H}$. Overall, we have for $\phi_y \in \mathcal{H}^*$

$$\psi(\phi_y) = \langle \phi_y, \phi_x \rangle_{\mathcal{H}^*} = \langle x, y \rangle_{\mathcal{H}} = \phi_y(x)$$

which shows surjectivity.

2. Denote by $\pi : \mathbb{R}^n \to \mathbb{T}^n$ the natural projection, which is continuous. We deduce that m is Haar measure from the fact that the Lebesgue measure $m_{\mathbb{R}^n}$ has this property (this is usually shown in a course on measure theory). Notice that by definition for any (Borel) measurable $A \subset \mathbb{T}^n$

$$m(A) = m_{\mathbb{R}^n}(\pi^{-1}(A) \cap [0,1]^n) = m_{\mathbb{R}^n}(\pi^{-1}(A) \cap [0,1)^n) = m_{\mathbb{R}^n}(\pi^{-1}(A) \cap (0,1)^n)$$

as the boundary of the hyper-cube $[0,1]^n$ is a null set for the Lebesgue measure $m_{\mathbb{R}^n}$.

We first need to show that any compact set $K \subset \mathbb{T}^n$ has finite measure. The set

$$\tilde{K} = \pi^{-1}(K) \cap [0,1]^n$$

is closed and thus compact as a subset of the compact set $[0,1]^n$. Note that it has image K. In particular,

$$m(K) = \int_{[0,1]^n} \chi_K \, \mathrm{d}m_{\mathbb{R}^n} = m_{\mathbb{R}^n}(\tilde{K}) < \infty.$$

where χ denotes the characteristic function.

Let $O \subset \mathbb{T}^n$ be open; we claim that $m(O) > 0$. For this, notice that

$$m(O) = m_{\mathbb{R}^n}(\pi^{-1}(O) \cap [0,1]^n) = m_{\mathbb{R}^n}(\pi^{-1}(O) \cap (0,1)^n) > 0$$

as $\pi^{-1}(O) \cap (0,1)^n$ is open.

Finally, let $A \subset \mathbb{T}^n$ be (Borel) measurable and let $a \in \mathbb{R}^n$. Then

$$m(a + A) = m_{\mathbb{R}^n}((a + \pi^{-1}(A)) \cap [0,1]^n) = m_{\mathbb{R}^n}(\pi^{-1}(A) \cap ([0,1]^n - a))$$

by translation invariance of the Lebesgue measure. Notice that we can write $\pi^{-1}(A) = \bigsqcup_{k \in \mathbb{Z}^n} A_k$ where $A_k = \pi^{-1}(A) \cap ([0,1)^n + k)$. As $A_0 = A_k - k$ translation invariance also implies

$$m(a + A) = \sum_{k \in \mathbb{Z}^n} m_{\mathbb{R}^n}(A_k \cap ([0,1]^n - a)) = \sum_{k \in \mathbb{Z}^n} m_{\mathbb{R}^n}(A_0 \cap ([0,1]^n - a - k))$$

As $\bigsqcup_{k \in \mathbb{Z}^n}([0,1)^n - a - k) = \mathbb{R}^n$ this implies $m(a + A) = m_{\mathbb{R}^n}(A_0) = m(A)$ as desired.

3. We prove first that W is closed. Since clearly, W is the kernel of the operator L, it suffices to show that L is indeed bounded. For this, we use the Cauchy-Schwarz-inequality to see that for any $x \in V$ (or actually $\ell^2(\mathbb{N})$)

$$\left| \sum_n \frac{x_n}{n} \right| \le \|x\|_2 \left(\sum_n \frac{1}{n^2} \right)^{\frac{1}{2}}$$

which proves boundedness.

Now if $v \in V$ is orthogonal to W, it must be orthogonal to $w^{(n)} \in W$ (for any $n \in \mathbb{N}$) where $w_1^{(n)} = 0$, $w_n^{(n)} = -n$ and $w_k^{(n)} = 0$ otherwise. In other words,

$$0 = \left\langle v, w^{(n)} \right\rangle = y_1 - ny_n$$

so that $v_n = \frac{1}{n}v_1$ for all $n \in \mathbb{N}$. However, as $v \in c_c(\mathbb{N}) = V$ there must be $n \in \mathbb{N}$ with $v_n = 0$. This shows that $v_1 = 0$ and hence $v = 0$.

For the only remaining claim we assume that there exists $v \in V$ with $L(x) = \langle x, v \rangle$ for all $x \in V$. This vector v must, as the kernel of L is W, be orthogonal to W and so $v = 0$. However, L is non-trivial (e.g. $L(1, 0, 0, \ldots) = 1$) which is a contradiction.

4. **a)** Fix $y \in \mathcal{H}$ and note that the map

$$\varphi_x : y \in \mathcal{H} \mapsto \overline{B(x, y)}$$

is linear and bounded, as

$$|\varphi_x(y)| = |\overline{B(x, y)}| = |B(x, y)| \leq M\|x\|\|y\|.$$

By the Frechet-Riesz theorem there exists a unique $Tx \in \mathcal{H}^*$ with $\overline{B(x, y)} = \langle y, Tx \rangle$ for all $y \in \mathcal{H}$ or equivalently

$$B(x, y) = \langle Tx, y \rangle \tag{1}$$

as required. This defines a map

$$T : \mathcal{H} \to \mathcal{H}$$

which is uniquely characterized by the property in (1).

We claim that T is linear. If $x_1, x_2 \in \mathcal{H}$ then for any $y \in \mathcal{H}$

$$\begin{aligned} \langle Tx_1 + Tx_2, y \rangle &= \langle Tx_1, y \rangle + \langle Tx_2, y \rangle = B(x_1, y) + B(x_2, y) \\ &= B(x_1 + x_2, y) \end{aligned}$$

The vector $Tx_1 + Tx_2$ thus satisfies the property required of $T(x_1 + x_2)$ and by uniqueness $T(x_1 + x_2) = Tx_1 + Tx_2$. One proceeds similarly for homogeneity and thus, T is linear.

To see that T is bounded, notice that for any $x \in \mathcal{H}$

$$\|Tx\|^2 = |\langle Tx, Tx \rangle| = |B(x, Tx)| \leq M\|x\|\|Tx\|$$

so that $\|Tx\| \leq M\|x\|$ as follows from division with $\|Tx\|$ (if $Tx = 0$ this is also clear).

b) The coercivitiy assumption yields for any $x \in \mathcal{H}$

$$c\|x\|^2 \leq |B(x,x)| = |\langle Tx, x\rangle| \leq \|Tx\|\|x\|$$

together with Cauchy-Schwarz or equivalently

$$c\|x\| \leq \|Tx\|. \tag{2}$$

We use Equation (2) to show that T is bijective. Observe first that T has closed image. In fact, by (2) if $(Tx_k)_k$ is a Cauchy-sequence, so is $(x_k)_k$ and hence the image of T is complete and thus closed. Let $y \in \mathcal{H}$ be orthogonal to the image of T. Then in particular,

$$0 = |\langle Ty, y\rangle| = |B(y,y)| \geq c\|y\|^2$$

and so $y = 0$. Corollary 3.17 on the orthogonal decomposition implies that T is surjective. Injectivity follows directly from (2).

We have thus a linear map $T^{-1} : \mathcal{H} \to \mathcal{H}$. By (2) applied for $x = T^{-1}(y)$ we have

$$c\|T^{-1}y\| \leq \|y\|$$

which proves also the required bound on the operator norm.

5. For measurable ψ the statement $\psi \in \mathcal{L}^2$ is equivalent to $|\psi| \in \mathcal{L}^2$. Also, if $\psi g \in \mathcal{L}^2$ for all $g \in \mathcal{L}^2$ then the same holds for $|\psi|$. This can be seen by considering the sign function

$$\operatorname{sign}(\psi)(x) = \begin{cases} \frac{\psi(x)}{|\psi(x)|} & \text{if } \psi(x) \neq 0 \\ 0 & \text{else} \end{cases}$$

which is in $\mathcal{L}^\infty$. Therefore, we may assume that ψ is a non-negative (real-valued) function.

We claim that the linear map

$$\Psi : g \in L^2 \mapsto \int_X \psi g \,\mathrm{d}\mu \in \mathbb{C}$$

is bounded (it is well-defined by assumption on ψ). Assume the contrary and find a sequence $g_k \in \mathcal{L}^2$ with $|\int_X \psi g_k \,\mathrm{d}\mu| > 3^k$ and $\|g_k\|_2 = 1$. Since

$$\left|\int_X \psi g_k \,\mathrm{d}\mu\right| \leq \int_X \psi |g_k| \,\mathrm{d}\mu$$

by non-negativity of ψ we may assume that $g_k \geq 0$ as well for all k (by replacing with $|g_k|$). Define

$$g = \sum_k 2^{-k} g_k$$

which is an element of $\mathcal{L}^2$ (as $\|g_k\|_2 = 1$). Plugging g into the definition of Ψ we obtain by monotone convergence

$$\int_X \psi g \,\mathrm{d}\mu = \sum_k 2^{-k} \int_X \psi g_k \,\mathrm{d}\mu \geq \sum_k \frac{3^k}{2^k}$$

which diverges and thus contradicts our assumption. Thus, Ψ is a bounded linear functional.

By the Frechet-Riesz representation theorem there exists $h \in \mathcal{L}^2$ with

$$\Psi(g) = \int_X hg \,\mathrm{d}\mu$$

for all $g \in L^2$. In particular, $\int_X (h - \psi) g \,\mathrm{d}\mu = 0$ for all $g \in L^2$. Replacing g with $\operatorname{sign}(h - \psi) g$ we obtain

$$\int_X |h - \psi| g \,\mathrm{d}\mu = 0$$

for all $g \in L^2$. Using this for characteristic functions of measurable sets we obtain that $|h - \psi| = 0$ almost everywhere. Thus, $\psi \in \mathcal{L}^2$.

6. For $v \in I$ we have $Uv = v$, hence $\frac{1}{N} \sum_{n=1}^{N} U^n v = \frac{1}{N} \sum_{n=1}^{N} v = v$. By definition of a projection P_I is the identity on I. So the convergence has to hold for this subspace.

Let us define $B := \{Uw - w : w \in \mathcal{H}\}$. Notice that $B \subset I^\perp$ as for any $Uw - w \in B$ and $v \in I^\perp$

$$\langle Uw - w, v \rangle = \langle Uw, v \rangle - \langle w, v \rangle = \langle w, Uv \rangle - \langle w, v \rangle = \langle w, v \rangle - \langle w, v \rangle = 0.$$

For $v = Uw - w \in B$ we have

$$\frac{1}{N} \sum_{n=1}^{N} U^n v = \frac{1}{N} \sum_{n=1}^{N} U^{n+1} w - U^n w = \frac{1}{N} (U^{N+1} w - Uw)$$

as $U^n v = U^{n+1} w - U^n w$. Hence the norm of this expression can be bounded by $\frac{1}{N} \|U^{N+1} w - Uw\| \leq \frac{1}{N} (\|w\| + \|w\|)$ which goes to 0 for $N \to \infty$. Since $Uw - w \in I^\perp$ its projection has to be 0, showing the claimed convergence on the subset $B \subset I^\perp$.

We claim that $B \subset I^\perp$ is dense. By Corollary 3.26 we may show that $B^\perp = I$ for which only one inclusion is still open. So assume $v \in B^\perp$. Then for all $w \in \mathcal{H}$ we have

$$0 = \langle v, Uw - w \rangle = \langle v, Uw \rangle - \langle v, w \rangle = \langle Uv, w \rangle - \langle v, w \rangle = \langle Uv - v, w \rangle$$

and applying this to $w = Uv - v$ yields $Uv - v = 0$ i.e. $v \in I$.

Now if $v \in I^\perp$ is arbitrary, we may choose for $\epsilon > 0$ some $v' \in B$ with $\|v - v'\| < \epsilon$. Then

$$\left\| \frac{1}{N} \sum_{n=1}^{N} U^n v \right\| = \left\| \frac{1}{N} \sum_{n=1}^{N} U^n (v - v') \right\| + \left\| \frac{1}{N} \sum_{n=1}^{N} U^n v' \right\| < \epsilon + \left\| \frac{1}{N} \sum_{n=1}^{N} U^n v' \right\|$$

and letting $N \to \infty$ we obtain

$$\limsup_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^{N} U^n v \right\| \leq \epsilon.$$

As $\epsilon > 0$ this proves the claim of the exercise for elements of $I^\perp$ and since any element of $\mathcal{H}$ can be written as a sum of an element of I and an element of $I^\perp$, we are done.