D-MATH Functional analysis I Fall 2018
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Solutions for exercise sheet 6

1. Write $f = \sum_{n\in\mathbb{Z}} a_n\chi_n$ where the functions χ_n are the characters of the 1-torus and are given by $\chi_n(x) = \mathrm{e}^{2\pi \mathrm{i} nx}$ for $n \in \mathbb{Z}$ and $x \in \mathbb{T}$. We know that $a_0 = 0$ as $\int_{\mathbb{T}} f(t)\,\mathrm{d}t = 0$. Then by Theorem 3.57 we have $f' = \sum_{n\in\mathbb{Z}}(2\pi\mathrm{i}n)a_n\chi_n$ and so

$$\|f\|_{L^2}^2 = \sum_{n\in\mathbb{Z}} |a_n|^2, \quad \|f'\|_{L^2}^2 = \sum_{n\in\mathbb{Z}} (2\pi)^2 n^2 |a_n|^2.$$

Since $a_0 = 0$, the inequality $|a_n|^2 \le n^2|a_n|^2$ holds for all $n \in \mathbb{Z}$ and hence

$$\|f\|_{L^2}^2 \le \tfrac{1}{(2\pi)^2} \sum_{n\in\mathbb{Z}} (2\pi)^2 n^2 |a_n|^2 = \tfrac{1}{(2\pi)^2}\|f'\|_{L^2}^2$$

as desired.

Now assume that $\|f\|_{L^2}^2 = \frac{1}{(2\pi)^2}\|f'\|_{L^2}^2$. This is equivalent to $|a_n|^2 = n^2|a_n|^2$ for all $n \in \mathbb{Z}$ and is true for $n = 1$ and $n = -1$. For $|n| > 1$ it implies $a_n = 0$ and thus concludes the exercise.

2. **a)** Note first that you can view any character χ_G on G as a character on $G \times H$ by extending it trivial i.e. by defining

$$\chi_G^{\mathrm{ext}}(g,h) = \chi_G(g)$$

for all $(g,h) \in G \times H$. The same holds for H. We claim that the homomorphism

$$\widehat{G} \times \widehat{H} \to \widehat{G \times H},\ (\chi_G, \chi_H) \mapsto \chi_G^{\mathrm{ext}}\chi_H^{\mathrm{ext}}$$

is an isomorphism. The map is injective: if $\chi_G^{\mathrm{ext}}\chi_H^{\mathrm{ext}}$ is the trivial character, then

$$\chi_G(g)\chi_H(h) = \chi_G^{\mathrm{ext}}(g,h)\chi_H^{\mathrm{ext}}(g,h) = 1$$

for all $(g,h) \in G \times H$. Applying this for $h = e_H$ yields that χ_G is trivial and applying it for $g = e_G$ yields that χ_H is trivial.

The map surjective: If χ is a character on $G \times H$ notice that

$$\chi(g,h) = \chi(g, e_H)\chi(e_G, h)$$

exactly write χ as a product of two characters, one on G and one H, as required.

b) As in the hint define for $\overline{k} = k + N\mathbb{Z} \in \mathbb{Z}/N\mathbb{Z}$

$$\chi_{\overline{k}} : \mathbb{Z}/N\mathbb{Z} \to \mathbb{S}^1,\ x + N\mathbb{Z} \mapsto \mathrm{e}^{2\pi \mathrm{i} xk/N}.$$

Notice that the division by N is what makes χ_k well-defined. One checks that χ_k is a character of $\mathbb{Z}/N\mathbb{Z}$. We claim that the map

$$\overline{k} \in \mathbb{Z}/N\mathbb{Z} \mapsto \chi_{\overline{k}} \in \widehat{\mathbb{Z}/N\mathbb{Z}}$$

is an isomorphism. It is injective: if $\chi_{\overline{k}}$ is trivial then in particular $\mathrm{e}^{2\pi \mathrm{i} k/N} = 1$ and hence $\frac{k}{N} \in \mathbb{Z}$ i.e. $\overline{k} = 0$.

To prove surjectivity, let[1] χ be an arbitrary character and define $\xi = \chi(\overline{1})$. Then ξ is an N-th root of unity as

$$\xi^N = \chi^N(\overline{1}) = \chi^N(\overline{N}) = \chi^N(\overline{0}) = 1.$$

and may hence be written as $\xi = \mathrm{e}^{2\pi \mathrm{i} k/N}$ for $k \in \{0, \ldots, N-1\}$. For any $\overline{x} \in \mathbb{Z}/N\mathbb{Z}$ we then have

$$\chi(\overline{x}) = \chi^x(\overline{1}) = \mathrm{e}^{2\pi \mathrm{i} kx/N}$$

so $\chi = \chi_{\overline{k}}$.

c) By the classification of finite abelian groups, G may be written as a product of groups $G_1, \ldots, G_m$ as in (b). By (a)

$$\widehat{G} \cong \widehat{G_1} \times \ldots \times \widehat{G_m} \cong G_1 \times \ldots \times G_m \cong G$$

where in the second step we used (b).

3. Define

$$G = \{(z_\gamma)_\gamma \in \mathbb{T}^\Gamma : z_{\gamma_1+\gamma_2} = z_{\gamma_1} + z_{\gamma_2} \text{ for all } \gamma_1, \gamma_2 \in \Gamma\}.$$

which is (as it is defined by equations) a closed subgroup of $\mathbb{T}^\Gamma$ and hence a compact abelian group. Note that checking that $\mathbb{T}^\Gamma$ is a metrizable topological group can be done as in the solution to 4a). The group G naturally identifies with the group of characters of Γ when Γ is equipped with the discrete topology.

It remains to find all characters of G. For $\gamma_0 \in \Gamma$ the map

$$\chi_{\gamma_0} : (z_\gamma)_\gamma \in G \mapsto \mathrm{e}^{2\pi \mathrm{i} z_{\gamma_0}}$$

[1] Alternatively, one can show that the characters $\chi_{\overline{k}}$ separate points.

defines a character of G. The character χ_{γ_0} is trivial if $z_{\gamma_0} = 0$ for all $(z_\gamma)_\gamma$ and thus $\gamma_0 = 0$ as otherwise this would contradict the theorem on completeness of characters (because G identifies with the group of characters of Γ). In other words, we have obtain a map

$$\gamma_0 \in \Gamma \mapsto \chi_{\gamma_0} \in \widehat{G}$$

which is in fact an injective homomorphism by definition of G.

It remains to show that any character χ on G can be written as $\chi = \chi_{\gamma_0}$ for some $\gamma_0 \in \Gamma$. By continuity of χ, $\chi^{-1}(B^{\mathbb{S}^1}_{1/10}(1))$ is open and contains the identity, so we may choose an open set U of the form $\prod_\gamma U_\gamma \cap G$ with $\chi(U) \subset B^{\mathbb{S}^1}_{1/10}(1)$ where all but finitely many U_γ are equal to $\mathbb{T}$. We list the exceptions as $U_{\gamma_1}, \ldots, U_{\gamma_d}$. Then U contains

$$H = \{(z_\gamma)_\gamma \in G : z_{\gamma_1} = \ldots = z_{\gamma_d} = 0\}$$

which is a closed subgroup of G. By definition of U, the character χ maps H to $B^{\mathbb{S}^1}_{1/10}(1)$. But then $\chi|_H$ must be trivial as otherwise any $z = (z_\gamma)_\gamma \in H$ with $\chi(z) \neq 1$ can be multiplied by $n \in \mathbb{Z}$ so that $\chi(nz) = \chi(z)^n$ lies outside of $B^{\mathbb{S}^1}_{1/10}(1)$.

The above shows that χ can be viewed as a character on the group G/H which we now interpret as a subgroup of $\mathbb{T}^d$. In fact, we have an injective continuous homomorphism

$$\Phi : G/H \ni (z_\gamma)_\gamma + H \mapsto (z_{\gamma_1}, \ldots, z_{\gamma_d}) \in \mathbb{T}^d$$

which induces therefore an isomorphism (and a homeomorphism) with the image G'. Thus, χ yields a character on G'.

Let us briefly describe the characters on G' and note that any character χ_n on $\mathbb{T}^d$ for $n \in \mathbb{Z}^d$ yields a character on G'. We claim that any character on G' is of this form. Indeed, consider the algebra $\mathcal{A}$ of finite linear combinations of the restrictions $\chi_n|_{G'}$. Since the characters χ_n separate points, so does $\mathcal{A}$ and hence as in the proof of Theorem 3.47 $\mathcal{A}$ is dense in $L^2(G')$. Any character on G' not of the form $\chi_n|_{G'}$ would need to be orthogonal to $\mathcal{A}$ and hence cannot exist.

We conclude that $\chi = \chi_n \circ \Phi$ for some $n \in \mathbb{Z}^d$. Therefore,

$$\chi\big((z_\gamma)_\gamma\big) = \mathrm{e}^{2\pi\mathrm{i}(n_1 z_{\gamma_1} + \ldots + n_d z_{\gamma_d})} = \mathrm{e}^{2\pi\mathrm{i} z_{\gamma_0}}$$

where $\gamma_0 = n_1\gamma_1 + \ldots + n_d\gamma_d$. This concludes the exercise.

4. **a)** By the already given explanations we only need to prove that the topology is metrizable and that addition is continuous.

The former follows directly from Exercise 1b), Sheet 0 where we proved that a countable product of metrizable spaces is metrizable. So the topology on the space $X = \prod_m \mathbb{Z}/p^m\mathbb{Z}$ is metrizable and hence the same holds for the induced topology on $\mathbb{Z}_p$. We remark that the standard metric on $\mathbb{Z}_p$ is given by

$$\mathrm{d}_p(a,b) = \begin{cases} p^{-\inf\{k \mid a_k \neq b_k\}+1} & \text{if } a \neq b \\ 0 & \text{else} \end{cases}$$

for $a = (a_m)_m$ and $b = (b_m)_m$ in $\mathbb{Z}_p$. One of the nice features of this metric is that $d_p(a,b) = d_p(a-b,0)$.

To prove that addition is continuous, it suffices to prove that the addition map $X \times X \to X$ is continuous. Let U be an open neighborhood of $a \in X$. Since the topology on X is the product topology and each of the factors in X is discrete, we may assume that U is given by

$$U = \{x \in X : x_m = a_m \text{ for all } m \in F\}$$

where $F \subset \mathbb{N}$ is a finite set. The preimage of U under the addition map is the set of points (x,y) with $x_m = -y_m - a_m$ for all $m \in F$. Let (x,y) be any such point. Letting U_1 be the set of points x' in X with $x'_m = x_m$ and defining U_2 analogously we obtain that the image of $U_1 \times U_2$ is in U as desired.

b) Let m be a Haar measure on $\mathbb{Z}_p$. By compactness we may assume that $m(\mathbb{Z}_p) = 1$ holds. We would first like to compute that Haar measure of any preimage of a points under π_k. Notice that

$$\mathbb{Z}_p = \bigsqcup_{x \in \mathbb{Z}/p^k\mathbb{Z}} \pi_k^{-1}(\{x\}). \tag{1}$$

For simplicity (and since it is customary) let us define $p^k\mathbb{Z}_p$ as the preimage of zero under π_k. If x' is any point in $\pi_k^{-1}(\{x\})$ for $x \in \mathbb{Z}/p^k\mathbb{Z}$ then we have

$$\pi_k^{-1}(\{x\}) = x' + \pi_k^{-1}(\{0\}) = x' + p^k\mathbb{Z}_p.$$

Left invariance of the Haar measure thus shows that

$$m(\pi_k^{-1}(\{x\})) = m(p^k\mathbb{Z}_p)$$

in this case. It is however easy to see that the projection maps are surjective. For instance, by viewing $\mathbb{Z}$ as a subset of Z_p via the map $x' \mapsto (\ldots, x', x', x')$.

Summing this up, if we take the Haar measure in (1) we obtain

$$1 = m(\mathbb{Z}_p) = \sum_{x \in \mathbb{Z}/p^k\mathbb{Z}} m(\pi_k^{-1}(\{x\})) = p^k m(p^k\mathbb{Z}_p).$$

Therefore, any preimage of a point under π_k has measure p^{-k}.

We claim that this property characterizes the measure uniquely. First of all, notice that any open set $O \subset \mathbb{Z}_p$ may be written as a countable disjoint union of sets of the form $x + p^k\mathbb{Z}_p$. By definition of the product topology it is certainly true that O may be written as a union of finite intersections of such sets. If there is $a \in (x+p^k\mathbb{Z}_p) \cap (y+p^\ell\mathbb{Z}_p)$ then for $k \geq \ell$ then $\pi_m(a-x) = a_m - x_m$ is zero for all $m \leq k$. Applying this to y we obtain $y_m = a_m$ for $m \leq \ell$ as well so $y_m = x_m$ for $m \leq \ell$. Thus, $(y + p^\ell\mathbb{Z}_p) \subset (x + p^k\mathbb{Z}_p)$ and O may be written in the desired fashion. The union is countable as $\mathbb{Z}_p$ is compact. Therefore, the measure of any open set is determined by $m(p^k\mathbb{Z}_p) = p^{-k}$ for all $k \in \mathbb{Z}$.

We now somewhat refine this argument. Note that the characteristic function of any set of the form $x + p^k\mathbb{Z}_p$ is continuous by (1). Let $\mathcal{A}$ be the set of finite linear combinations of such characteristic functions. By a previous verification $\mathcal{A}$ is an algebra and is thus dense in $C(\mathbb{Z}_p)$. Density of $C(\mathbb{Z}_p)$ in $L^1(\mathbb{Z}_p)$ as well as $\|\cdot\|_1 \leq \|\cdot\|_\infty$ implies that $\mathcal{A} \subset L^1(\mathbb{Z}_p)$ is dense. Therefore, the integral of any integrable function on $\mathbb{Z}_p$ is determined by the integrals of functions in $\mathcal{A}$ and hence by the equality $m(p^k\mathbb{Z}_p) = p^{-k}$ for all $k \in \mathbb{Z}$.

c) For any character χ_k on $\mathbb{Z}/p^k\mathbb{Z}$ we can define a character on $\mathbb{Z}_p$ by $\chi_k \circ \pi_k$. Note that characters of this form separate points: if $x \neq y \in \mathbb{Z}_p$ there is a k with $x_k \neq y_k$. The characters of $\mathbb{Z}/p^k\mathbb{Z}$ separate points so there is χ_k with $\chi_k(x_k) \neq \chi_k(y_k)$. In other words, $\chi = \chi_k \circ \pi_k$ satisfies $\chi(x) \neq \chi(y)$ as desired.

5. To simplify notation, we let χ_m denote the character of the torus associated to $m \in \mathbb{Z}$.

a) First, assume that the sequence is equidistributing. Then for any $m \in \mathbb{Z}$

$$\tfrac{1}{n}\sum_{k=0}^{n-1} \chi_m(x_k) \to \int_{\mathbb{T}} \chi_m(t)\,\mathrm{d}t$$

as $n \to \infty$. By orthogonality of characters, the integral is zero whenever χ_m is non-trivial i.e. $m \neq 0$.

We first prove the converse implication for functions in $C(\mathbb{T})$. For this, observe that if f is a (finite) trigonometric sum of the form $\sum_m' c_m\chi_m$ the convergence

$$\tfrac{1}{n}\sum_{k=0}^{n-1} f(x_k) = \sum_m{}' c_m \tfrac{1}{n}\sum_{k=0}^{n-1} \chi_m(x_k) \to c_0 = \int_{\mathbb{T}} f(t)\,\mathrm{d}t$$

holds. Otherwise, notice that the algebra $\mathcal{A}$ of (finite) trigonometric polynomials is dense in $C(\mathbb{T})$ by the Theorem of Stone-Weierstrass. Therefore, if $f \in C(\mathbb{T})$

is arbitrary and $\epsilon > 0$, let $g \in \mathcal{A}$ with $\|f - g\|_\infty < \epsilon$ and choose $N \in \mathbb{N}$ large enough such that

$$\left| \tfrac{1}{n} \sum_{k=0}^{n-1} g(x_k) - \int_{\mathbb{T}} g(t)\,\mathrm{d}t \right| < \epsilon.$$

Then

$$\begin{aligned}
&\left| \tfrac{1}{n} \sum_{k=0}^{n-1} f(x_k) - \int_{\mathbb{T}} f(t)\,\mathrm{d}t \right| \\
&\leq \left| \tfrac{1}{n} \sum_{k=0}^{n-1} f(x_k) - \tfrac{1}{n} \sum_{k=0}^{n-1} g(x_k) \right| + \left| \tfrac{1}{n} \sum_{k=0}^{n-1} g(x_k) - \int_{\mathbb{T}} g(t)\,\mathrm{d}t \right| \\
&\quad + \left| \int_{\mathbb{T}} g(t)\,\mathrm{d}t - \int_{\mathbb{T}} f(t)\,\mathrm{d}t \right| \\
&< \tfrac{1}{n} \sum_{k=0}^{n-1} |f(x_k) - g(x_k)| + \epsilon + \int_{\mathbb{T}} |f(t) - g(t)|\,\mathrm{d}t < 3\epsilon
\end{aligned}$$

as desired. The claim is thus proven for functions in $C(\mathbb{T})$.

For the general case, let $f \in C([0,1])$ and assume without loss of generality that f is real-valued (otherwise, use this for the real and the imaginary part) and that $f(0) \leq f(1)$ (otherwise, consider $-f$). Let $\delta > 0$ be small and define a function $f_+ \in C(\mathbb{T})$ as follows: set $f_+(x) = f(x)$ when $x \geq \delta$. Otherwise, define $x_\delta < \delta$ as the smallest point in $[0,1]$ where the linear interpolation between $(\delta, f(\delta))$ and $(0, f(1))$ intersects the graph of f. Using this, we set $f_+(x) = f(x)$ for $x \geq x_\delta$ and otherwise let $f_+(x)$ be given by the linear interpolation. Then $f \leq f_+$ and by choosing δ small enough we can ensure that

$$\int_0^1 f_+(t) - f(t)\,\mathrm{d}t < \epsilon$$

for some ϵ. Similarly, one construct $f_- \leq f$ with $f_-(1) = f(0)$, $f_- \leq f$ and

$$\int_0^1 f(t) - f_-(t)\,\mathrm{d}t < \epsilon$$

Applying the equidistribution statement to f_+, f_- one obtains

$$\begin{aligned}
\limsup_{n\to\infty} \tfrac{1}{n} \sum_{k=0}^{n-1} f(x_k) &\leq \int_{[0,1]} f_+(t)\,\mathrm{d}t \leq \int_{\mathbb{T}} f(t)\,\mathrm{d}t + \epsilon \\
\liminf_{n\to\infty} \tfrac{1}{n} \sum_{k=0}^{n-1} f(x_k) &\geq \int_{[0,1]} f_-(t)\,\mathrm{d}t \geq \int_{\mathbb{T}} f(t)\,\mathrm{d}t - \epsilon
\end{aligned}$$

from which the equidistribution statement for f follows.

b) Notice first that the leading digit of 2^n is k if and only if

$$k \cdot 10^m \leq 2^n < (k+1)10^m$$

for some $m \in \mathbb{N}_0$. Taking logarithms, this is equivalent to

$$\log(k) + m \leq n \log_{10}(2) < \log(k+1) + m$$

or in other words $\log(k) \leq \{n\alpha\} < \log(k+1)$ where $\alpha = \log_{10}(2)$.

Note that α is irrational. Indeed, if $\log_{10}(2) = \frac{p}{q}$ for some rational number $\frac{p}{q}$ then $2^q = 10^p$. Since 5^p divides the right hand side, it divides the left hand side so $p = 0$ and thus $q = 0$ which is impossible.

Let f be the characteristic function of the interval $I_k = [\log_{10}(k), \log_{10}(k+1))$. Since f is not continuous we cannot apply a) directly, but need to approximate f from above and below by continuous functions. Note that the class of functions where this can be done is exactly the set of Riemann-integrable functions. Pick for $\epsilon > 0$ continuous functions $f_+, f_- : [0,1] \to \mathbb{R}$ with

$$f_- \leq f \leq f_+, \quad \int_{[0,1]} f_+(t) - f_-(t) \, \mathrm{d}t < \epsilon.$$

Such functions can for instance be picked to be piecewise linear. The desired statement for f then follows as in a).

6. a) We should first explain what we mean by a C^k-function on $\mathbb{T}^d$. We identify any function $f : \mathbb{T}^d \to \mathbb{C}$ with a $\mathbb{Z}^d$-periodic function $\mathbb{R}^d \to \mathbb{C}$ also denoted by f. A function $f : \mathbb{T}^d \to \mathbb{C}$ is C^k if its counterpart $f : \mathbb{R}^d \to \mathbb{C}$ is C^k. So we may naturally view $C^k(\mathbb{T}_d) \subset C^k_b(\mathbb{R}^d)$ where the latter is a Banach space by the referenced example. The norm on $C^k_b(\mathbb{R}^d)$ has the property that is describe also in the exercise.

It remains to show that $C^k(\mathbb{T}_d) \subset C^k_b(\mathbb{R}^d)$ is closed. For this, let $f_k \to f$ where the functions $f_k \in C^k_b(\mathbb{R}^d)$ are $\mathbb{Z}^d$-periodic and $f \in C^k_b(\mathbb{R}^d)$. Note that periodicity of the functions f_k implies periodicity of the all of their derivatives. We need to show that f is $\mathbb{Z}^d$-periodic. Clearly, we have pointwise convergence $f_k(x) \to f(x)$ and so for any $n \in \mathbb{Z}^d$ and $x \in \mathbb{R}^d$ taking limits in the equality $f_k(x) = f_k(x+n)$ implies the desired statement.

b) Let $f \in C^\infty(\mathbb{T}^d)$ and write

$$f = \sum_{n \in \mathbb{Z}^d} a_n(f) \chi_n.$$

Then we can apply Theorem 3.57 to any derivative $\partial_\alpha f$ of f to obtain that

$$\|\partial_\alpha f\|_\infty \leq \sum_{n\in\mathbb{Z}} |a_n(\partial_\alpha f)| = \sum_{n\in\mathbb{Z}} (2\pi)^{\|\alpha\|_1} |n^\alpha| |a_n(f)| < \infty$$

where $n^\alpha = n_1^{\alpha_1} \cdots n_d^{\alpha_d}$. Applying this to all α of the form $n_1^{2\ell_1} \cdots n_d^{2\ell_d}$ with $\ell_1 + \ldots + \ell_d = \ell$ we obtain by summing using the multinomial theorem

$$\sum_{n\in\mathbb{Z}} \|n\|_2^{2\ell} |a_n(f)| < \infty.$$

In particular, $\sum_{n\in\mathbb{Z}} (1 + \|n\|_2^{2\ell}) |a_n(f)| < \infty$ and thus $(1 + \|n\|_2^{2\ell}) |a_n(f)|$ is a bounded sequence. This proves the first direction.

Now let $f : \mathbb{T}^d \to \mathbb{C}$ be square-integrable with the property that

$$|a_n(f)| \ll_\ell \frac{1}{1 + \|n\|_2^{2\ell}}$$

for all $\ell \in \mathbb{N}$. We first claim that

$$\sum_{n\in\mathbb{Z}^d} (2\pi)^{\|\alpha\|_1} |n^\alpha| |a_n(f)| < \infty \tag{2}$$

for any multiindex α. Surely, for $n \in \mathbb{Z}^d$ fixed and $j \in \{1, \ldots, d\}$ we have using the binomial theorem

$$\begin{aligned} |n_j|^{\alpha_j} \leq (1 + n_j^2)^{\alpha_j} \leq (1 + \|n\|_2^2)^{\alpha_j} &= \sum_{k=0}^{\alpha_j} \binom{\alpha_j}{k} \|n\|_2^{2k} \\ &\leq \sum_{k=0}^{\alpha_j} \binom{\alpha_j}{k} (1 + \|n\|_2^{2\alpha_j}) = 2^{\alpha_j} (1 + \|n\|_2^{2\alpha_j}). \end{aligned}$$

Using this same technique, the inequality

$$1 + x^k \leq (1 + x)^k \ll 1 + x^k \tag{3}$$

follows for $x \geq 0$. Thus, by taking products

$$|n^\alpha| \ll_\alpha \prod_j (1 + \|n\|_2^{2\alpha_j}) \ll_d (1 + \|n\|_2^{2\|\alpha\|_\infty})^d \ll 1 + \|n\|_2^{2\|\alpha\|_\infty d}.$$

Let $\ell \in \mathbb{N}$ be arbitrary (to be determined later) then applying our assumption

$$\sum_{n\in\mathbb{Z}^d} (2\pi)^{\|\alpha\|_1} |n^\alpha| |a_n(f)| \ll_\alpha \sum_{n\in\mathbb{Z}^d} \frac{1 + \|n\|_2^{2\|\alpha\|_\infty d}}{1 + \|n\|_2^{2\ell}}$$

and choosing $\ell = k\|\alpha\|_\infty d$ we obtain by (3)

$$\sum_{n\in\mathbb{Z}^d}(2\pi)^{\|\alpha\|_1}|n^\alpha||a_n(f)| \ll_\alpha \sum_{n\in\mathbb{Z}^d}\frac{1+\|n\|_2^{2\|\alpha\|_\infty d}}{(1+\|n\|_2^{2\|\alpha\|_\infty d})^k}$$
$$= \sum_{n\in\mathbb{Z}^d}\frac{1}{(1+\|n\|_2^{2\|\alpha\|_\infty d})^{k-1}}$$
$$\leq \sum_{n\in\mathbb{Z}^d}\frac{1}{1+\|n\|_2^{2\|\alpha\|_\infty d(k-1)}}.$$

For k large enough, this sum is convergent (see Equation (3.22) for a more precise statement). This proves the claim in (2).

This implies that the smooth function

$$f_N = \sum_{n\in\mathbb{Z}^d:\|n\|_\infty\leq N} a_n(f)\chi_n$$

form a Cauchy-sequence in $C^k(\mathbb{T}^d)$ (where k is arbitrary). Indeed, for $N > M$ and a multiindex α we have

$$\|\partial_\alpha(f_N - f_M)\|_\infty \leq \sum_{n\in\mathbb{Z}:\|n\|_\infty>M}(2\pi)^{\|\alpha\|_1}|n^\alpha||a_n(f)| \to 0$$

as $M \to \infty$ by (2). Thus, by a) the sequence $(f_N)_N$ is convergent. Since k is arbitrary, the limit must be a smooth function $\tilde{f} \in C^\infty(\mathbb{T}^d)$. Since uniform convergence implies L^2-convergence, the functions f_N also converge to $\tilde{f}$ in L^2, but this must imply that $f = \tilde{f}$ as claimed.

c) By b) it suffices to show that

$$|a_n(f)| \ll_\ell \frac{1}{1+\|n\|_2^{2\ell}}$$

for all $\ell \in \mathbb{N}$. In fact, since the partial derivatives $\partial_j^k f$ exist and are continuous, the partial integration in Section 3.4.3 still applies and we have

$$a_n(\partial_j^k f) = (2\pi \mathrm{i} n_j)^k a_n(f).$$

Therefore, since $\partial_j^k f \in L^2(\mathbb{T}^d)$

$$\sum_{n\in\mathbb{Z}}|a_n(\partial_j^k f)|^2 = \sum_{n\in\mathbb{Z}}(2\pi)^{2k}|n_j|^{2k}|a_n(f)|^2 < \infty$$

Summing over j we obtain

$$\sum_{n\in\mathbb{Z}}\|n\|_2^{2k}|a_n(f)|^2 < \infty.$$

By adding one and using the fact that any sequence with convergent series must be bounded we deduce for even k

$$|a_(f)| \ll \frac{1}{\sqrt{1+\|n\|_2^{2k}}} \leq \frac{1}{1+\|n\|_2^k}$$

as desired.