Prof. Dr. Manfred Einsiedler
Andreas Wieser

Solutions for exercise sheet 7

1. Let m_G be a Haar measure on G. We compute

$$\begin{aligned}\langle v, f *_\pi w\rangle &= \int_G \langle v, f(g)\pi_g w\rangle \,\mathrm{d}m_G(g) = \int_G \overline{f(g)}\,\langle v, \pi_g w\rangle \,\mathrm{d}m_G(g)\\ &= \int_G \overline{f(g)}\,\langle \pi_g^{-1} v, w\rangle \,\mathrm{d}m_G(g) = \int_G \overline{f(g)\langle w, \pi_g^{-1} v\rangle}\,\mathrm{d}m_G(g).\end{aligned}$$

The first equality uses the definition of the weak integral (see Proposition 3.83) and the convolution (see Definition 3.85) and third uses that π_g is unitary. Pulling the conjugation out of the integral we may as well compute

$$\begin{aligned}\int_G f(g)\,\langle w, \pi_g^{-1} v\rangle \,\mathrm{d}m_G(g) &= \int_G f(g)\,\langle w, \pi_{g^{-1}} v\rangle \,\mathrm{d}m_G(g)\\ &= \int_G f(g^{-1})\,\langle w, \pi_g v\rangle \,\mathrm{d}m_G(g)\\ &= \int_G \langle w, f^*(g)\pi_g v\rangle \,\mathrm{d}m_G(g) = \langle w, f^* *_\pi v\rangle\end{aligned}$$

where we used that the substitution $\iota : g \in G \mapsto g^{-1} \in G$ preserves the Haar measure. Indeed, one verifies the measure $\iota_* m_G$ defined by $\iota_* m_G(B) = m_G(\iota^{-1}(B))$ for measurable sets B is also a left Haar measure (since G is abelian). By the uniqueness properties of Haar measure there exists $\lambda > 0$ such that $\iota_* m_G = \lambda m_G$. But then

$$m_G = \iota_*(\iota_* m_G) = \iota_*(\lambda m_G) = \lambda \iota_* m_G = \lambda^2 m_G$$

and so $\lambda = 1$. This shows that $\iota_* m_G = m_G$ and yields as in (3.23) for any integrable function ψ

$$\int_G \psi(g)\,\mathrm{d}m_G(g) = \int_G \psi\circ\iota^{-1}(g)\,\mathrm{d}m_G = \int_G \psi(g^{-1})\,\mathrm{d}m_G(g).$$

Summing up, we have

$$\langle v, f *_\pi w\rangle = \overline{\langle w, f^* *_\pi v\rangle} = \langle f^* *_\pi v, w\rangle$$

as desired.

2. Let $x = (x_1, \dots, x_n, 0, 0, \dots) \in V$. We have

$$\|T_1 x\| \le \dots \le \|T_n x\| = \|T_{n+1} x\| = \dots,$$

and therefore $\{T_n x : n \in \mathbb{N}\}$ is bounded.

On the other hand, define $x^{(n)} := (1, \dots, 1, 0, 0, \dots)$ to be the sequence which is 1 in the first n coordinates and 0 afterwards. Then $\|x^{(n)}\| = 1$ for all n. But

$$\|T_n x^{(n)}\| = \|(1, 2, \dots, n, 0, 0, \dots)\| = n,$$

so that $\|T_n\|_{\text{op}} \ge n$. Thus we have $\sup_{n\in\mathbb{N}} \|T_n\|_{\text{op}} = \infty$ as claimed.

3. **a)** It is clear that $\mathcal{H}' := \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$ contains 0 and is closed under scalar multiplication. Moreover, the inequality $\|v_n + w_n\|^2_{\mathcal{H}_n} \le 2(\|v_n\|^2_{\mathcal{H}_n} + \|w_n\|^2_{\mathcal{H}_n})$ implies that $\mathcal{H}'$ is a vector space.

The fact that $\langle \cdot, \cdot \rangle$ defines an inner product is an immediate consequence of the $\langle \cdot, \cdot \rangle_{\mathcal{H}_n}$ being inner products.

To show that $\mathcal{H}'$ is complete, let $(x^{(n)})_n \subseteq \mathcal{H}'$ be a Cauchy sequence. Let $\epsilon > 0$ be given, and let $N \in \mathbb{N}$ be such that for $n_1, n_2 \ge N$ we have

$$\sum_{m\ge 1} \|x_m^{(n_1)} - x_m^{(n_2)}\|^2_{\mathcal{H}_m} = \|x^{(n_1)} - x^{(n_2)}\|^2 < \epsilon.$$

In particular, $(x_m^{(n)})_n$ is a Cauchy sequence in $\mathcal{H}_m$ for any m and thus converges. Let $x_m \in \mathcal{H}_m$ be its limit and let $x = (x_m)_m$. We claim that $x \in \mathcal{H}'$ and that $x^{(n)} \to x$ as $x \to \infty$. For $n \ge N$ we have

$$\begin{aligned}
\sum_{m\ge 1} \|x_m^{(n)} - x_m\|^2_{\mathcal{H}_m} &= \sup_{M\ge 1} \sum_{m\le M} \|x_m^{(n)} - x_m\|^2_{\mathcal{H}_m} \\
&= \sup_{M\ge 1} \lim_{n'\to\infty} \sum_{m\le M} \|x_m^{(n)} - x_m^{(n')}\|^2_{\mathcal{H}_m} \\
&\le \limsup_{n'\to\infty} \|x^{(n)} - x^{(n')}\|^2 < \epsilon.
\end{aligned}$$

Since

$$\|x\|^2 \le 2(\|x - x^{(n)}\|^2 + \|x^{(n)}\|^2) < \infty,$$

the assertion follows.

b) Define the map

$$\begin{aligned}
\Phi : \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n &\to \overline{\langle \mathcal{H}_n \rangle} \\
(v_n)_n &\mapsto \lim_{N\to\infty} \sum_{n\le N} v_n.
\end{aligned}$$

Firstly, this is a well-defined map: Let $\epsilon > 0$ and let $N \in \mathbb{N}$ be such that $\sum_{n>N} \|v_n\|^2_{\mathcal{H}_n} < \epsilon$. Then for $N_1 \geq N_2 > N$ we have

$$\| \sum_{n \leq N_1} v_n - \sum_{n \leq N_2} v_n \|^2 \leq \sum_{n > N_2} \|v_n\|^2 < \epsilon.$$

Hence, $(\sum_{n \leq N} v_n)_N$ defines a Cauchy sequence, and since $\overline{\langle \mathcal{H}_n \rangle}$ is a Hilbert space, it converges.

It is clear that Φ is linear. To see that it is isometric, let $v = (v_n)_n, w = (w_n)_n \in \bigoplus_{n \in \mathbb{N}} \mathcal{H}_n$. Since inner products are continuous (Exercise 1, Sheet 4), we have

$$\begin{aligned}\langle \Phi(v), \Phi(w) \rangle &= \langle \lim_{N \to \infty} \sum_{n \leq N} v_n, \lim_{M \to \infty} \sum_{m \leq M} w_m \rangle \\ &= \lim_{N,M \to \infty} \sum_{n \leq N} \sum_{m \leq M} \langle v_n, w_m \rangle = \sum_{n \geq 1} \langle v_n, w_n \rangle_{\mathcal{H}_n} = \langle v, w \rangle.\end{aligned}$$

Lastly, we have to show that Φ is surjective. To this end, let $v \in \overline{\langle \mathcal{H}_n \rangle}$. Then

$$\lim_{N \to \infty} d(v, \bigoplus_{n \leq N} \mathcal{H}_n) = 0.$$

By Corollary 3.18, the orthogonal projection of v onto $\bigoplus_{n \leq N} \mathcal{H}_n$ is its best approximation. Hence, denoting by $\pi_n : \mathcal{H} \to \mathcal{H}_n$ the projection map, we have that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|v - \sum_{n \leq N} \pi_n v\|^2 = \|v - \pi_{\bigoplus_{n \leq N} \mathcal{H}_n} v\|^2 < \epsilon.$$

This implies $v = \lim_{N \to \infty} \sum_{n \leq N} \pi_n v$. Note also that $(\pi_n v)_n \in \bigoplus_{n \in \mathbb{N}} \mathcal{H}_n$ since

$$\sum_n \|\pi_n v\|^2 = \lim_{N \to \infty} \left\| \sum_{n \leq N} \pi_n v \right\|^2 = \|v\|^2.$$

Hence, $\Phi((\pi_n v)_n) = v$.

4. **a)** We first construct a character using the hint. Let us note that H is indeed normal as $sHs^{-1} = H$ (since $srs^{-1} = r^{-1}$) so that and character on D_3/H yields a character on D_3 by precomposition with the projection $D_3 \to D_3/H$.

Note that H has index two in D_3 (the quotient D_3/H is generated by sH) so that there are two characters on D_3/H, namely the trivial character and a non-trivial character. Let χ_s be the precomposition of this character with the projection above. Then $\chi_s(s) = -1$. So we have two characters on D_3 and want to know if we can possibly find more.

So let χ be an arbitrary character on D_3. Since $\chi(s)^2 = \chi(s^2) = 1$ we must have either $\chi(s) = 1$ or $\chi(s) = -1$. In the latter case we can replace χ by $\chi\chi_s^{-1}$ and may thus assume that $\chi(s) = 1$. Then

$$\chi(r^{-1}) = \chi(srs^{-1}) = \chi(s)\chi(r)\chi(s^{-1}) = \chi(r)$$

so $\chi(r)^2 = \chi(r^2) = 1$. This implies that

$$\chi(r) = \chi(rr^3) = \chi(r^2)^2 = 1$$

and shows that χ is trivial. Thus, the trivial character and χ_s are the full set of characters on D_3.

These do not separate points as for instance $\chi_s(r) = 1$ and so there is no character that separates 1 from r.

b) Theorem 3.80 implies that

$$\mathcal{H} = \bigoplus_{\chi'} \mathcal{H}_{\chi'}$$

where χ' runs over all characters of H. Since $\mathbb{Z}/3\mathbb{Z} \simeq H$ and $\chi^2 \neq 1$, $1, \chi, \chi^2$ are all characters of H and so we obtain the desired decomposition.

To give an elementary argument (without referring to Theorem 3.80) let us write down an explicit decomposition. For this, let $v \in \mathcal{H}$ be arbitrary and define

$$\begin{aligned}
v_0 &= v + \pi_r v + \pi_{r^2} v \\
v_1 &= v + \xi \pi_r v + \xi^2 \pi_{r^2} v \\
v_2 &= v + \xi^2 \pi_r v + \xi^4 \pi_{r^2} v
\end{aligned}$$

where ξ is a non-trivial third root of unity such as $\xi = \mathrm{e}^{2\pi \mathrm{i}/3}$. Then

$$\begin{aligned}
\pi_r v_0 &= \pi_r v + \pi_{r^2} v + \pi_{r^3} v = \pi_r v + \pi_{r^2} v + v = v_0 \\
\pi_r v_1 &= \pi_r v + \xi \pi_{r^2} v + \xi^2 \pi_{r^3} v = \xi^{-1} v_1 \\
\pi_r v_2 &= \pi_r v + \xi^2 \pi_{r^2} v + \xi^4 \pi_{r^3} v = \xi v_2
\end{aligned}$$

so $v_0 \in \mathcal{H}_1$, $v_1 \in \mathcal{H}_{\chi^2}$ and $v_2 \in \mathcal{H}_\chi$. The matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi & \xi^2 \\ 1 & \xi^2 & \xi^4 \end{pmatrix}$$

has determinant $(\xi - 1)(\xi^2 - 1)(\xi^2 - \xi)$, is invertible and hence v can be written as a linear combination of v_0, v_1, v_2. Of course, one can write it down explicitly:

$$v = \tfrac{1}{3}(v_0 + v_1 + v_2)$$

c) Let $v \in \mathcal{H}_1$ (or equivalently $\pi_r v = v$). To show that $\pi_s v \in \mathcal{H}_1$ we compute

$$\pi_r(\pi_s v) = \pi_s \pi_{srs} v = \pi_s \pi_{r^{-1}} v = \pi_s v.$$

Let $v \in \mathcal{H}_\chi$. Then

$$\pi_r(\pi_s v) = \pi_s \pi_{srs} v = \pi_s \pi_{r^{-1}} v = \chi(r^{-1}) \pi_s v = \chi^2(r) \pi_s v$$

and so $\pi_s v \in \mathcal{H}_{\chi^2}$.

Let $v \in \mathcal{H}_{\chi^2}$. Then

$$\pi_r(\pi_s v) = \pi_s \pi_{srs} v = \pi_s \pi_{r^{-1}} v = \chi^2(r^{-1}) \pi_s v = \chi(r) \pi_s v$$

as desired.

5. We would like to define a continuous injective linear map

$$\iota : \ell^1(\mathbb{Z}) \to C(\mathbb{T}).$$

For this, we define for any $a \in \ell^1(\mathbb{Z})$

$$\iota(a) = \sum_{n \in \mathbb{Z}} a_n \chi_n.$$

We first need to show that the right hand side is absolutely convergent, which then implies that $\iota(a) \in C(\mathbb{T})$ as $C(\mathbb{T})$ is a Banach space. Indeed, for any $a \in \ell^1(\mathbb{Z})$ the triangle inequality implies

$$\sum_{n \in \mathbb{Z}} |a_n| \|\chi_n\|_\infty = \sum_{n \in \mathbb{Z}} |a_n| = \|a\|_1 < \infty \tag{1}$$

and thus ι is well-defined. It follows also from the definition that ι is linear. Furthermore, the inequality

$$\|\iota(a)\|_\infty \leq \sum_{n \in \mathbb{Z}} |a_n| \|\chi_n\|_\infty = \|a\|_1$$

as in (1) implies that ι is bounded and thus linear. Injectivity follows from the fact that for any $a \in \ell^1(\mathbb{Z})$ the series $\sum_{n \in \mathbb{Z}} a_n \chi_n$ converges in L^2 (as $\|\cdot\|_2 \leq \|\cdot\|_\infty$) which yields that the coefficients are uniquely determined (see Theorem 3.54). In fact, they are given by $a_n = \langle \iota(a), \chi_n \rangle$ for all $n \in \mathbb{Z}$ which shows that for any a, a' with $\iota(a) = \iota(a')$, $a_n = a'_n$ holds for all $n \in \mathbb{Z}$ i.e. $a = a'$.

It remains to show that ι satisfies

$$\iota(a)\iota(b) = \iota(a * b)$$

for all $a, b \in \ell^1(\mathbb{Z})$. Indeed,

$$\left(\sum_{n\in\mathbb{Z}} a_n\chi_n\right)\left(\sum_{m\in\mathbb{Z}} b_m\chi_m\right) = \sum_{m,n\in\mathbb{Z}} a_n b_m \chi_{n+m} = \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} a_n b_{k-n}\chi_k$$

as desired where we used absolute convergence to interchange sums.

6. **a)** Fix $n \in \mathbb{N}$ and assume that $f \in C(\mathbb{R}^2)$. Define the function

$$g_n : v \in \mathbb{R}^2 \mapsto \int_{\mathbb{T}} \overline{\chi_n}(\vartheta) f(k_\vartheta^{-1}.v)\,\mathrm{d}\vartheta$$

Using uniform continuity of the function f in large balls it is straightforward to show that g_n is continuous. We now claim that $g_n = f_n$. For this, it suffices to prove that g_n satisfies the defining property of f_n (see Proposition 3.83). So let $\phi \in L^2(\mathbb{R}^2)$ and compute

$$\begin{aligned}
\langle \phi, g_n\rangle_{L^2} &= \int_{\mathbb{R}^2} \phi(v)\overline{g_n(v)}\,\mathrm{d}v = \int_{\mathbb{R}^2}\int_{\mathbb{T}} \phi(v)\chi_n(\vartheta)\overline{f(k_\vartheta^{-1}.v)}\,\mathrm{d}\vartheta\,\mathrm{d}v \\
&= \int_{\mathbb{T}} \chi_n(\vartheta)\int_{\mathbb{R}^2} \phi(v)\overline{\pi_{k_\vartheta} f(v)}\,\mathrm{d}v\,\mathrm{d}\vartheta = \int_{\mathbb{T}} \chi_n(\vartheta)\,\langle \phi, \pi_{k_\vartheta} f\rangle\,\mathrm{d}\vartheta \\
&= \int_{\mathbb{T}} \langle \phi, \overline{\chi_n}(\vartheta)\pi_{k_\vartheta} f\rangle\,\mathrm{d}\vartheta.
\end{aligned}$$

By the uniqueness in Proposition 3.83 we deduce that $g_n = \overline{\chi_n} * f = f_n$ in L^2.

b) For $v \in \mathbb{R}^2$, let $F_v \in C^1(\mathbb{T})$ be defined by

$$F_v(\vartheta) = f(k_\vartheta^{-1}v).$$

Then

$$f_n(v) = \int_{\mathbb{T}} \overline{\chi_n(\vartheta)} f(k_\vartheta^{-1}v)d\vartheta = \langle F_v, \chi_n\rangle_{L^2(\mathbb{T})} = a_n(F_v).$$

Theorem 3.57 therefore implies

$$\sum_{n\in\mathbb{Z}} |f_n(v)| = \sum_{n\in\mathbb{Z}} |a_n(F_v)| \ll \sqrt{\|F_v\|_2^2 + \|F_v'\|_2^2}.$$

Now

$$\|F_v\|_2^2 = \int_{\mathbb{T}} |f(k_\vartheta^{-1}v)|^2 d\vartheta \leq \|f\|^2_{\infty, B_{\|v\|}(0)}$$

and from the chain rule we infer that

$$\begin{aligned}
\|F_v'\|_2^2 &= \int_{\mathbb{T}} \left|\left\langle \nabla f(k_\vartheta^{-1}v), \frac{\partial}{\partial\vartheta}(k_\vartheta^{-1}v)\right\rangle_{\mathbb{R}^2}\right|^2 d\vartheta \\
&\ll \max\{\|\partial_1 f\|_{\infty, B_{\|v\|}(0)}, \|\partial_2 f\|_{\infty, B_{\|v\|}(0)}\}\|v\|
\end{aligned}$$

using the Cauchy-Schwarz inequality.

In particular, for any $R > 0$ we have

$$\sup_{v \in B_R(0)} \sum_{n \in \mathbb{Z}} |f_n(v)| < \infty$$

and the claim follows.