Solutions for exercise sheet 8

1. By the closed graph theorem (Theorem 4.28) it suffices to show that A is a closed operator as A is defined everywhere. So assume that $(x_k, Ax_k) \in \mathrm{graph}(A)$ is given for $k \in \mathbb{N}$ and that $(x_k, Ax_k) \to (x, y) \in \mathcal{H} \times \mathcal{H}$. The topology on $\mathcal{H} \times \mathcal{H}$ is the product topolgy here and could for instance be induced by the maximum norm (see Sheet 1). We need to show that $y = Ax$. For any $z \in \mathcal{H}$ we have by continuity of the inner product

$$\langle Ax_k, z \rangle \to \langle y, z \rangle$$

and on the other hand by self-adjointness

$$\langle Ax_k, z \rangle = \langle x_k, Az \rangle \to \langle x, Az \rangle = \langle Ax, z \rangle .$$

By uniqueness of limits, this shows that $\langle y, z \rangle = \langle Ax, z \rangle$ and thus $y = Ax$ as z was arbitrary. (What this is saying, is that y and Ax determine the same continuous linear functional and so must be equal by injectivity in Frechet-Riesz. Alternatively, one can apply the statement to the difference $y - Ax = z$ and deduce that $\|y - Ax\|^2 = 0$.)

2. The statement will be a consequence of the Baire category theorem. Proceeding by contradiction, assume that there are $x_1, x_2, \ldots$ in X which form a basis of X. For every $n \in \mathbb{N}$ define

$$V_n = \mathrm{span}(x_1, \ldots, x_n).$$

By assumption, any element x of X may be written as a linear combination of finitely many x_n's and if k is the biggest index appearing in this linear combination, $x \in V_k$. Thus,

$$\bigcup_{n \in \mathbb{N}} V_n = X.$$

Notice that the $V_n's$ are closed (they are finite-dimensional and thus complete) and have empty interior. Indeed, if there is $\epsilon > 0$ with $B_\epsilon(v) \subset V_n$ for some $v \in V_n$, then $B_\epsilon \subset V_n - v = V_n$. If $x \in X \setminus \{0\}$ is arbitrary then $x\frac{\epsilon}{\|x\|} \in B_\epsilon \subset V_n$ or in other words, $x = \frac{\|x\|}{\epsilon} v$ for some $v \in V_n$ and so $x \in V_n$.

Therefore, the subspace V_n are nowhere dense and hence X is meagre. This contradicts completeness of X by the Baire category theorem.

3. **a)** One immediately checks that $Y^\perp$ is a subspace: if $x_1^*, x_2^* \in Y^\perp$ and λ is a scalar, then for any $y \in Y$

$$(x_1^* + \lambda x_2^*)(y) = x_1^*(y) + \lambda x_2^*(y) = 0.$$

To show that $Y^\perp$ is closed, let $(x_k^*)_k$ be a sequence of elements in the annihilator and assume that $x_k^* \to x^* \in X^*$. Note that convergence in the operator norm implies pointwise convergence. Thus, for any $y \in Y$

$$|x^*(y)| = |x_k^*(y) - x^*(y)| \to 0$$

as $k \to \infty$ which implies $x^*(y) = 0$. As y was arbitrary, $x^* \in Y^\perp$ as desired.

b) We may assume that Y is closed. In fact, a continuity argument shows $\overline{Y}^\perp = Y^\perp$ and $\inf_{y \in Y} \|x - y\| = \inf_{y \in \overline{Y}} \|x - y\|$ as Y is dense in $\overline{Y}$.

Let $x \in X$. One equality can be obtained by elementary means: suppose that $x^* \in Y^\perp$ with $\|x^*\| \leq 1$. Then for any $y \in Y$

$$|x^*(x)| = |x^*(x) - x^*(y)| \leq \|x^*\| \|x - y\| \leq \|x - y\|$$

and so

$$|x^*(x)| \leq \inf_{y \in Y} \|x - y\|.$$

For the converse inequality, we need to construct an element of the annihilator. Note that there is nothing to show if $x \in Y$ and so we assume $x \in X \setminus Y$. Define a linear map

$$z^* : ax + y \in Z = \operatorname{span}(x) \oplus Y \mapsto a \inf_{y \in Y} \|x + y\|.$$

Arguing as in Corollary 7.6 this defines a bounded linear functional on Z.

Choose a Hahn-Banach extension $x^* \in X^*$ of z^* (see Theorem 7.3). By construction, $x^*|_Y = z^*|_Y = 0$ or in other words, $x^* \in Y^\perp$. Also,

$$\inf_{y \in Y} \|x + y\| = |x^*(x)|$$

as desired.

c) We consider the natural map

$$\Phi : Y^\perp \to (X/Y)^*, x^* \mapsto \overline{x}^*$$

where $\overline{x}^*$ is defined by $\overline{x}^*(x + Y) = x^*(x)$ and is well-defined as $x^* \in Y^\perp$. Φ is bijective: in fact, an inverse to Φ is given by

$$\overline{x}^* \in (X/Y)^* \mapsto x^* = \overline{x}^* \circ \pi \in Y^\perp$$

where π denotes the projection $X \to X/Y$.

It remains to show that Φ is an isometry. Since π is 1-Lipschitz, $\|x^*\| \le \|\overline{x}^*\|$ by submultiplicativity of the operator norm. For the converse inequality, note that the 1-ball in X/Y is given by the cosets $x + Y$ where x can be chosen to be in the 1-ball of X. Thus,

$$\sup_{x+Y \in X/Y : \|x\| \le 1} |\overline{x}^*(x+Y)| = \sup_{x+Y \in X/Y : \|x\| \le 1} |x^*(x)| = \|x^*\|$$

as claimed.

4. We follow the hint and let $\{x_n^* : n \in \mathbb{N}\}$ be a countable dense subset of X^*. By definition of the operator norm we can in fact choose $x_n \in X$ with norm at most 1 and with

$$|x_n^*(x_n)| \ge \tfrac{\|x_n^*\|}{2}$$

for every $n \in \mathbb{N}$. Define Y as the closure of the $\mathbb{Q}$-linear span of the x_n's. By definition, Y contains the $\mathbb{Q}$-linear span of the x_n's as a dense subset and is hence separable. We show that $X = Y$.

Assume by contradiction that $X \neq Y$. The proof of Corollary 7.6 shows that for any given $x_0 \in X \setminus Y$ there is $x^* \in X^*$ with $x^*|_Y = 0$ and $x^*(x_0) = 1$.

By density of the x_n^*'s we may choose n_0 such that

$$\|x^* - x_{n_0}^*\| < \epsilon$$

for some $\epsilon > 0$. This would also imply that

$$|x_{n_0}^*(x_{n_0})| = |x^*(x_{n_0}) - x_{n_0}^*(x_{n_0})| < \varepsilon$$

since x_{n_0} is inside the unit ball. On the other hand,

$$|x_{n_0}^*(x_{n_0})| \ge \tfrac{\|x_n^*\|}{2} \ge \tfrac{\|x^*\| - \epsilon}{2}$$

and both of these inequalities cannot be true for all $\epsilon > 0$. This is a contradiction.

5. a) Let us first show that ϕ_p is surjective. So let $f \in \ell^p(\mathbb{N})^*$. We need to find $x \in \ell^q(\mathbb{N})$ such that $f = \phi_p(x)$. Clearly, the desired x satisfies

$$f(e^{(i)}) = \phi_p(x)(e_i) = x_i$$

for all $i \in \mathbb{N}$ where $e^{(i)}$ denotes the sequence which is 1 at i and zero otherwise. We thus define x as the sequence $(f(e^{(i)}))_i$ and first show that $x \in \ell^q(\mathbb{N})$. For this, fix $N \in \mathbb{N}$ and compute

$$\sum_{n=1}^{N} |x_n|^q = \sum_{n=1}^{N} |x_n||x_n|^{q-1} = \sum_{n=1}^{N} x_n \vartheta_n |x_n|^{q-1} = f(a) = |f(a)|$$

where ϑ_n is such that $x_n \vartheta_n = |x_n|$ for every $n \in \mathbb{N}$ and where $a \in c_c(\mathbb{N}) \subset \ell^p(\mathbb{N})$ is the sequence with $a_n = \vartheta_n |x_n|^{q-1}$ for $n \leq N$ and $a_n = 0$ for $n > N$. By continuity of f,

$$|f(a)| \leq \|f\| \|a\|_p = \|f\| \left(\sum_{n=1}^{N} |x_n|^q \right)^p$$

Putting things together yields by division $\|x\|_q \leq \|f\|$ when taking the limit $N \to \infty$ and in particular $x \in \ell^q(\mathbb{N})$. By definition $f = \phi_p(x)$ on finite linear combinations of the $e^{(i)}$'s i.e. on $c_c(\mathbb{N})$ and hence also on the whole space $\ell^p(\mathbb{N})$ by uniqueness of continuation and density of $c_c(\mathbb{N})$. The above proof also shows that $\|\phi_p(x)\| \geq \|x\|_q$ and so concludes the claim.

b) We again show that ϕ_∞ is a surjective isometry and begin with surjectivity. So let $f \in c_0(\mathbb{N})^*$ and define $x_n = f(e^{(n)})$. Then for any $N \in \mathbb{N}$

$$\sum_{n=1}^{N} |x_n| = \sum_{n=1}^{N} \vartheta_n x_n = f(a) = |f(a)| \leq \|f\| \|a\|_\infty = \|f\|$$

where ϑ_n is such that $x_n \vartheta_n = |x_n|$ for every $n \in \mathbb{N}$ and where $a \in c_c(\mathbb{N}) \subset c_0(\mathbb{N})$ is the sequence with $a_n = \vartheta_n$ for $n \leq N$ and $a_n = 0$ for $n > N$. Thus, taking the limit $N \to \infty$ shows that $\|x\|_1 \leq \|f\|$ and in particular $x \in \ell^1(\mathbb{N})$. Again, f and $\phi_\infty(x)$ coincide on $c_c(\mathbb{N})$ and thus $f = \phi_\infty(x)$.

It remains to show that $\|\phi_\infty(x)\| \leq \|x\|_1$. For this, simply note that for any $a \in c_0(\mathbb{N})$

$$|\phi_\infty(x)(a)| = \left| \sum_{k=1}^{\infty} x_k a_k \right| \leq \|a\|_\infty \sum_{k=1}^{\infty} |x_k|.$$

c) Applying[1] b) and a) (to $\ell^1(\mathbb{N})^*$) we have isometric isomorphisms

$$c_0(\mathbb{N})^{**} \simeq \ell^1(\mathbb{N})^* \simeq \ell^\infty(\mathbb{N}).$$

[1] The isomorphism between the duals is given by precomposition.

To show that the image of $c_0(\mathbb{N})$ in $c_0(\mathbb{N})^{**}$ under the natural embedding ι is not everything, we may as well determine the image in $\ell^\infty(\mathbb{N})$. Let $x \in c_0(\mathbb{N})$. As mentioned, the first isomorphism above is given by

$$f \in c_0(\mathbb{N})^{**} \mapsto f \circ \phi_\infty \in \ell^1(\mathbb{N})^*.$$

To view the way $c_0(\mathbb{N})$ embeds into $\ell^1(\mathbb{N})$: If $g \in c_0(\mathbb{N})^*$ then $\iota(x)(g) = g(x)$ and so for $a \in \ell^1(\mathbb{N})$

$$\iota(x) \circ \phi_\infty(a) = \phi_\infty(a)(x) = \sum_{k=1}^{\infty} a_k x_k.$$

Thus, the induced embedding $\iota' : c_0(\mathbb{N}) \to \ell^1(\mathbb{N})^*$ coincides with the isometric isomorphism $\ell^\infty(\mathbb{N}) \to \ell^1(\mathbb{N})^*$. In particular, since $c_0(\mathbb{N})$ is a proper subspace of $\ell^\infty(\mathbb{N})$, ι' and thus also ι are not surjective.

6. **a)** Let Y be a subspace of X and let $y^* \in Y^*$. Assume that $x_1^*, x_2^* \in X^*$ are two Hahn-Banach extensions as in Theorem 7.3. If $y^* = 0$ then $\|x_1^*\| = \|x_2^*\|$ and so $x_1^* = 0 = x_2^*$.

We may thus assume that $y^* \neq 0$. Replacing y^* by $\frac{y^*}{\|y^*\|}$ (and correspondigly x_i^* by $\frac{x_i^*}{\|y^*\|}$ for $i = 1, 2$) we may also suppose that

$$\|y^*\| = \|x_1^*\| = \|x_2^*\| = 1.$$

If x_1^* and x_2^* are distinct, strict convexity implies

$$\left\| \frac{x_1^* + x_2^*}{2} \right\| < \frac{\|x_1^*\|}{2} + \frac{\|x_2^*\|}{2} = 1$$

But $x^* = \frac{x_1^* + x_2^*}{2}$ also satisfies $x^*|_Y = y|_*$ and so

$$\|y^*\| \leq \|x^*\| < 1$$

which is a contradiction.

b) Consider the subspace of constant functions Y of $C([0,1], \|\cdot\|_\infty)$ and the functional $f : a \in Y \mapsto a$. The evaluation map $\mathrm{ev}_x \in C([0,1])^*$ at any point $x \in [0,1]$ would extend f. Note that all such evaluation maps have norm one and are distinct.