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Solutions for exercise sheet 9

1. Assume first that X is relexive and denote by $\iota : X \to X^{**}$ and by $\iota^* : X^* \to X^{***}$ the natural isometric embeddings. We first relate ι and ι^*. If $\ell \in X^{**}$ write $\ell = \iota(x)$ for $x \in X$ and obtain for any $x^* \in X^*$

$$\iota^*(x^*)(\ell) = \ell(x^*) = \iota(x)(x^*) = x^*(x) = x^* \circ \iota^{-1}(\ell)$$

In other words, $\iota^* : x^* \mapsto x^* \circ \iota^{-1}$ which is surjective as the inverse is given by precomposition with ι.

Now assume that X^* is reflexive and let $Y = \iota(X) \subset X^{**}$. As ι is isometric, Y is complete and thus closed. Suppose that $Y \neq X^{**}$ and let $x^{**} \in X^{**} \setminus Y$. By Corollary 7.6 there exists $\ell \in X^{***}$ such that

$$\ell|_Y = 0, \quad \ell(x^{**}) \neq 0.$$

As X^* is reflexive, we can write $\ell = \iota^*(x^*)$ for some $x^* \in X^*$. Then for any $x \in X$

$$0 = \ell(\iota(x)) = \iota(x)(x^*) = x^*(x).$$

But this implies that $x^* = 0$ (it vanishes identically on X) and thus $\ell = 0$). This is a contradiction to $\ell(x^{**}) \neq 0$ and hence $Y = X$ and ι is surjective.

2. We define for any $n \in \mathbb{N}$ the half-open interval $I_n = (\frac{1}{n+1}, \frac{1}{n}]$ so that

$$[0,1] = \bigsqcup_{n \in \mathbb{N}} I_n \cup \{0\}$$

As $[0,1]$ has finite measure, functions in $L^\infty_m([0,1])$ are integrable. We can thus define a linear map

$$\Phi : L^\infty_m([0,1]) \to \ell^\infty(\mathbb{N}), \quad f \mapsto \Big(\tfrac{1}{|I_1|} \int_{I_1} f \, \mathrm{d}m, \tfrac{1}{|I_2|} \int_{I_2} f \, \mathrm{d}m, \ldots \Big)$$

by averaging over the intervals I_n where $|I|$ denotes the length of an interval I. Notice that for any $f \in L^\infty_m([0,1])$ and any $n \in \mathbb{N}$ we have $\left| \int_{I_n} f \, \mathrm{d}m \right| \leq \|f\|_\infty$ and so $\|\Phi(f)\|_\infty \leq \|f\|_\infty$. That is, Φ is bounded of norm at most one (and in fact equal to

1). Let LIM be the Banach limit on $\ell^\infty(\mathbb{N})$ and define the bounded linear functional $\ell = \mathrm{LIM} \circ \Phi$.

We claim that ℓ cannot be represented by an L^1-function as required in the exercise. By contradiction, suppose that

$$\ell(f) = \int_{[0,1]} fg \,\mathrm{d}m$$

for all $f \in L^\infty_m([0,1])$ and some $g \in L^1_m([0,1])$. Consider the bounded operator (isometry) $\Psi : \ell^\infty(\mathbb{N}) \to L^\infty_m([0,1])$ which maps a sequence $a = (a_n)_n$ to the piecewise constant function f with $f|_{I_n} = a_n$ for all n. Then $\Phi \circ \Psi$ is the identity map and $\mathrm{LIM} = \ell \circ \Psi$ satisfies

$$\mathrm{LIM}(a) = \int_{[0,1]} \Psi(a) g \,\mathrm{d}m = \sum_{n\in\mathbb{N}} \int_{I_n} \Psi(a) g \,\mathrm{d}m = \sum_{n\in\mathbb{N}} a_n \int_{I_n} g \,\mathrm{d}m.$$

for all $a \in \ell^\infty(\mathbb{N})$. Notice that the sequence $b = (b_n)_n$ defined by $b_n = \int_{I_n} g \,\mathrm{d}m$ for all n satisfies

$$\sum_{n\in\mathbb{N}} |b_n| \leq \sum_{n\in\mathbb{N}} \int_{I_n} |g| \,\mathrm{d}m = \|g\|_1 < \infty$$

and thus lies in $\ell^1(\mathbb{N})$.

We have thus shown that the Banach limit is represented by an element $b \in \ell^1(\mathbb{N})$. This however cannot be true. For instance, if $e^{(i)}$ denotes the sequence which is 1 at the i-th coordinate and zero otherwise we have by invariance of the Banach limit (see Corollary 7.14)

$$b_1 = \mathrm{LIM}(e^{(1)}) = \mathrm{LIM}(e^{(2)}) = b_2 = \dots.$$

Thus, b is constant and must hence by identically zero. The Banach limit however is not identically zero, which is a contradiction.

3. For any function $f \in \ell^\infty(\mathbb{Z}^n)$ and any $m \in \mathbb{N}$ we define the average

$$f_m = \tfrac{1}{|I_m|} \sum_{x\in I_m} f(x)$$

where $I_m = [-m, m]^n \cap \mathbb{Z}^n$. By definition

$$|f_m| \leq \tfrac{1}{|I_m|} \sum_{x\in I_m} |f(x)| \leq \|f\|_\infty.$$

We have thus defined a bounded linear operator

$$\mathrm{av} : \ell^\infty(\mathbb{Z}^n) \to \ell^\infty(\mathbb{N}), \quad f \mapsto (f_m)_m$$

In fact, the norm is one and not only at most one as can be checked on the constant functions.

Let $L \in \ell^\infty(\mathbb{N})^*$ be an extension of the limit function on $c(\mathbb{N})$ as in the proof of Corollary 7.14 and define

$$\mathrm{LIM} = L \circ \mathrm{av} \in \ell^\infty(\mathbb{Z}^n)^*.$$

By submultiplicativity of norms LIM has norm at most one and in fact equal to one as can be checked on constant functions. It remains to check the properties (i) and (ii):

(i) If $f \in \ell^\infty(\mathbb{Z}^n)$ is non-negative then $\liminf_{m\to\infty} f_m \geq 0$ and thus $\mathrm{LIM}(f) \geq 0$ as follows from the corresponding property of the Banach-limit (see Corollary 7.14, the second item).

(ii) Let $k \in \mathbb{Z}^n$ be fixed and denote by f^k the map $x \in \mathbb{Z}^n \mapsto f(x-k)$ as defined on the sheet. Then for any $m \in \mathbb{N}$

$$\begin{aligned} |f_m - f_m^k| &= \left| \tfrac{1}{|I_m|} \sum_{x \in [-m,m]^n \cap \mathbb{Z}^n} f(x) - \tfrac{1}{|I_m|} \sum_{x \in ([-m,m]^n - k) \cap \mathbb{Z}^n} f(x) \right| \\ &\leq \tfrac{1}{|I_m|} \sum_{x \in B_m} |f(x)| \leq \|f\|_\infty \tfrac{|B_m|}{|I_m|} \end{aligned}$$

where B_m is the set of points that lie either(!) in the cube $[-m,m]^n \cap \mathbb{Z}^n$ or in the cube $([-m,m]^n - k) \cap \mathbb{Z}^n$. If we can show that $\frac{|B_m|}{|I_m|} \to 0$ as $m \to \infty$ then $\lim_{m\to\infty} f_m = \lim_{m\to\infty} f_m^k$ if the limit exists which shows that $\mathrm{LIM}(f) = \mathrm{LIM}(f^k)$ as desired.
To prove the remaining claim, let us estimate the size of

$$B'_m = (([-m,m]^n - k) \setminus [-m,m]^n) \cap \mathbb{Z}^n$$

. Notice that this set is contained in the rectanguler annulus

$$[-\|k\|_\infty - m, \|k\|_\infty + m]^n \setminus [-m,m]^n$$

which has Lebesgue measure

$$\ll (2m + 2\|k\|_\infty)^{n-1}(m + \|k_\infty\| - m) \ll (m + \|k\|_\infty)^{n-1} \ll m^{n-1}$$

Thus, $\frac{|B'_m|}{|I_m|} \ll \frac{1}{m}$ which goes to zero. Applying this argument twice (symmetrically) we obtain the desired claim.

4. **a)** The restriction of functionals in X^* to the subspace Y yields a bounded operator $\phi' : X^* \to Y^*$. By the Hahn-Banach theorem this operator is surjective and has norm one. By definition its kernel is equal to $Y^\perp$ and hence we obtain a bijective linear map

$$\phi : X^*/Y^\perp \to Y^*.$$

It remains to show that ϕ is an isometry. Let $x^* \in X^*$ and let $\epsilon > 0$. Choose $x_0^* \in Y^\perp$ so that

$$\|x^* + Y^\perp\|_{X^*/Y^\perp} \leq \|x^* + x_0^*\| \leq \|x^* + Y^\perp\|_{X^*/Y^\perp} + \epsilon.$$

Then

$$\begin{aligned}\|\phi(x^* + Y^\perp)\| &= \|x^*|_Y\| = \|(x^* + x_0^*)|_Y\| \leq \|x^* + x_0^*\| \\ &\leq \|x^* + Y^\perp\|_{X^*/Y^\perp} + \epsilon\end{aligned}$$

and $\|\phi(x^*+Y^\perp)\| \leq \|x^*+Y^\perp\|_{X^*/Y^\perp}$ since $\epsilon > 0$ was arbitrary. For the converse inequality, note that the Hahn-Banach theorem allows us to extend $y^* = \phi(x^* + Y^\perp)$ to an element $\tilde{x}^*$ preserving the norm. Replacing x^* by $\tilde{x}^*$ this shows the claim.

b) Let $\iota_X : X \to X^{**}$ be the natural embedding and similarly define ι_Y. Note that as X is reflexive, X is a Banach space and thus Y is also a Banach space as it is closed. We need to show that any element in Y^{**} can be represented by an element of Y. So let $\ell \in Y^{**}$. By a) we may identify ℓ with an element of the dual of $X^*/Y^\perp$ and by composing with the projection we obtain in particular an element of the dual of X^*. By reflexivity of X we may write the latter as $\iota(x)$ for $x \in X$. Note that $\iota(x)$ needs to vanish on elements of $Y^\perp$, that is,

$$\iota(x)(x^*) = x^*(x) = 0$$

for all $x^* \in Y^\perp$. By Exercise 3b), Sheet 8, x is an element of Y as Y is closed.

We claim that $\iota_Y(x) = \ell$. Let $y^* \in Y^*$ and choose $x^* \in X^*$ extending y^* as in the Hahn-Banach theorem (see a)). By the definitions we made

$$\ell(y^*) = \iota(x)(x^*) = x^*(x) = y^*(x)$$

as $x \in Y$. This concludes the claim.

c) The Banach space $\ell^\infty(\mathbb{N})$ is not reflexive as the subspace $c_0(\mathbb{N})$ is not reflexive by Exercise 5, Sheet 8.

Now let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space with no atoms. We recall that the latter means that for any $A \in \mathcal{B}$ of positive measure there exists $B \in \mathcal{B}$ with

$0 < \mu(B) < \mu(A)$. Write $X = \bigsqcup_{n\in\mathbb{N}} X_n$ for measurable subsets X_n of positive finite measure. As in Exercise 2 define an operator

$$\Psi : \ell^\infty(\mathbb{N}) \to L^\infty_\mu(X)$$

by sending a sequence a to the measurable function f with $f|_{X_n} = a_n$. This is an isometry and in particular, the image of Ψ is complete (thus closed) and not reflexive. Hence, $L^\infty_\mu(X)$ is not reflexive.

5. **a)** For $f \in L^1_\mu(X)$ and $n \in \mathbb{N}$ let A_n be the set of points $x \in X$ where $|f(x)| > \frac{1}{n}$. Then

$$\mu(A_n)\frac{1}{n} \leq \int_X |f(x)| \, \mathrm{d}\mu < \infty$$

and so A_n has finite measure. This implies that A_n is finite. Now observe that

$$\{x \in X : f(x) \neq 0\} = \bigcup_{n\in\mathbb{N}} A_n$$

is countable as desired.

b) Let $f : X \to \mathbb{C}$ be measurable. We may assume that f is real-valued: If the claim holds for $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ then it also holds for f. Furthermore, we may assume that f is bounded: Writing

$$X = \bigcup_{n\in\mathbb{N}} \{x \in X : |f(x)| \leq n\}$$

we see that there must be some n such that $\{x \in X : |f(x)| \leq n\}$ is cocountable (i.e. the complement of a countable set) as otherwise X would be countable. If the claim holds for the bounded measurable function $f' = \max(f, n)$ then it holds for f.

So let $f : X \to [-M, M]$ be measurable. We proceed inductively to construct the set A. Consider first the measurable set $\{x \in X : f(x) \geq 0\}$ which is by definition of the σ-algebra either countable of cocountable (i.e. the complement of a countable set). Define

$$I_1 = \begin{cases} [0, M] & \text{if } \{x \in X : f(x) \geq 0\} \text{ is cocountable} \\ [-M, 0] & \text{else} \end{cases}$$

To proceed for any closed interval $I \subset \mathbb{R}$ denote by $I^\ell = I \cap (-\infty, \mathrm{mid}(I)]$ the left half of the interval where $\mathrm{mid}(I)$ is the midpoint of I, and similarly $I^r = I \cap [\mathrm{mid}(I), \infty)$. Then define recursively

$$I_{n+1} = \begin{cases} I_n^r & \text{if } \{x \in X : f(x) \in I_n^r\} \text{ is cocountable} \\ I_n^\ell & \text{else} \end{cases}.$$

By construction (and completeness of $\mathbb{R}$) the interval $I = \bigcap_n I_n$ consists of a point and

$$A := \{x \in X : f(x) \notin I\} = \bigcup_{n \in \mathbb{N}} \{x \in X : f(x) \notin I_n\}$$

is countable. This proves the claim.

c) Define the operator

$$\phi : B(X) \to L^1_\mu(X)^*$$

by setting $\phi(g) : f \in L^1_\mu(X) \mapsto \sum_{x \in X} f(x)g(x)$ for $g \in B(X)$. Notice that the latter sum is over countably many points by a) and is absolutely convergent as

$$\left| \sum_{x \in X} f(x)g(x) \right| \leq \sum_{x \in X} |f(x)||g(x)| \leq \|g\|_\infty \|f\|_1.$$

for all $f \in L^1_\mu(X)$. In particular, $\|\phi(g)\| \leq \|g\|_\infty$. We first show that ϕ is an isometry. For $\epsilon > 0$ let $x_0 \in X$ such that $\|g\|_\infty \leq |g(x_0)| + \epsilon$ and let δ_{x_0} be the L^1-function which is 1 at x_0 and zero otherwise. Notice that $\|\delta_{x_0}\|_1 = 1$. Then

$$|\phi(g)(\delta_{x_0})| = |g(x_0)| \geq \|g\|_\infty - \epsilon$$

and so $\|\phi(g)\| \geq \|g\|_\infty - \epsilon$ for any $\epsilon > 0$. Thus, ϕ is an isometry.

It remains to show that ϕ is onto. So let $\ell \in L^1_\mu(X)^*$ and set $g(x) = \ell(\delta_x)$ for any $x \in X$. The function g is bounded as

$$|g(x)| \leq \|\ell\| \|\delta_x\|_1 = \|\ell\|.$$

By linear combinations one sees that $\phi(g)$ and ℓ are equal on functions in $L^1_\mu(X)$ of finite support. However, notice that there are dense in $L^1_\mu(X)$: if $f \in L^1_\mu(X)$ choose an enumeration $\{x_1, x_2, \ldots\} = \{x \in X : f(x) \neq 0\}$ and note that as the sum $\sum_{n=1}^\infty |f(x_n)|$ converges there is for any $\epsilon > 0$ some $N \in \mathbb{N}$ such that $\sum_{n=N}^\infty |f(x_n)| < \epsilon$. Thus, f is ϵ-close to $f_N = f(x_1)\delta_{x_1} + \ldots + f(x_N)\delta_{x_N}$. By density $\phi(g)$ and ℓ most be equal on $L^1_\mu(X)$.

6. Pick a normalized basis $\{v_n\}_n$ of the finite dimensional subspace V and let $\{v_n^*\}_n$ be the corresponding dual basis, so we have $v_m^*(v_n) = \delta_{mn}$ for all $1 \leq m, n \leq \dim(V)$. These are continuous functionals of norm one on V and we may extend them to linear functionals on all of X. For simplicity, let us denote these again by $v_1^*, v_2^*, \ldots$. Now let $w \in X$ be arbitrary and consider

$$w' = w - \sum_{n=1}^{\dim(V)} v_n^*(w) v_n.$$

Then for any m

$$v_m^*(w') = v_m^*(w) - \sum_{n=1}^{\dim(V)} v_n^*(w)v_m^*(v_n) = 0.$$

This shows that w can be written as a linear combination of an element in V and an element in the closed subspace

$$W = \{w' \in X : v_m^*(w') = 0 \text{ for all } m\}.$$

Since $W' \cap V = \{0\}$ by there is a unique such linear combination, which concludes the exercise.