D-MATH Functional analysis I Fall 2018
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Solutions for exercise sheet 10

1. First of all, note that since X is finite dimensional, any linear functional on X is continuous. Let $e_1, \ldots, e_n$ be a basis of X and choose elements $e_i^* \in X^*$ such that

$$e_i^*(e_j) = \delta_{ij}$$

for all $i, j \in \{1, \ldots, n\}$.

Let us now prove that the weak topology is equal to the norm topology on X. In general (without assuming that X is finite dimensional), the norm topology on X is finer than the weak topology. This follows from the fact that any element of X^* is continuous in the norm topology (by definition of X^*) and that the weak topology is the smallest (weakest) topology for which all elements of X^* are continuous.

Let us thus show that the norm topology on X is weaker than the weak topology. For this, we first let $\epsilon > 0$ and find a weakly neighborhood of $0 \in B_\epsilon(0)$. Notice that for any $x = \sum_{i=1}^n a_i e_i$ we have

$$\|x\| \leq \left\| \sum_{i=1}^n a_i e_i \right\| \leq \sum_{i=1}^n |a_i| \|e_i\| \leq C \max_{i=1,\ldots,n} |a_i| = C \max_{i=1,\ldots,n} |e_i^*(x)|$$

where $C = \sum_{i=1}^n \|e_i\|$. Therefore,

$$N_{e_1^*,\ldots,e_n^*;\epsilon/C}(0) \subset B_\epsilon(0). \tag{1}$$

Now let $U \subset X$ be open in the norm topology and let $x_0 \in U$. Choose $\epsilon > 0$ with $B_\epsilon(x) \subset U$. By (1) we have

$$N_{e_1^*,\ldots,e_n^*;\epsilon/C}(x_0) = x_0 + N_{e_1^*,\ldots,e_n^*;\epsilon/C}(0) \subset x_0 + B_\epsilon(0) = B_\epsilon(x_0) \subset U.$$

Thus, U is also open in the weak topology.

2. Let X be a σ-compact locally compact metric space and choose compact sets K_n such that

$$K_1 \subset K_2^\circ \subset K_2 \subset K_3^\circ \subset \ldots.$$

Let Λ be a positive linear functional on $C_c(X)$. By restriction we may view Λ as a positive linear functional on $C_c(K_n^\circ)$. Urysohn's lemma and the Hahn-Banach lemma then imply that $\Lambda|_{C_c(K_n^\circ)}$ can be extended to a linear functional on $C(K_n)$. By the Riesz representation theorem (compact case) there is a Borel measure on K_n representing Λ_n. Restricting again to $C_c(K_n^\circ)$ one obtains a Borel measure μ_n with

$$\Lambda(f) = \int_{K_n^\circ} f \,\mathrm{d}\mu_n$$

for every $f \in C_c(K_n^\circ)$. From the uniqueness in the compact case of Riesz representation theorem it follows that $\mu_{n+1}|_{K_n^\circ} = \mu_n$ for every n. Thus, one can "glue" these measures together to obtain a measure μ on X that has the property constructed above for functions $f \in C_c(K_n^\circ)$ for every n. Since any compactly supported function has support in some K_n° this concludes the (sketch of) proof.

3. **a)** Let $(x_n)_n$ be a weakly convergent sequence in X and let x be the limit. We will use the Banach-Steinhaus theorem to show that $(x_n)_n$ is bounded. For this, consider $\iota(x_n)$ for every n where $\iota : X \to X^{**}$ is the canonical isometric embedding. By assumption, for any $x^* \in X^*$

$$\iota(x_n)(x^*) = x^*(x_n) \to x^*(x)$$

for $n \to \infty$. In particular, $\|\iota(x_n)(x^*)\|$ is bounded. This shows that the operators $\iota(x_n)$ are pointwise bounded. By the Banach-Steinhaus theorem applied to X^* and these pointwise bounded operators

$$\sup_{n\in\mathbb{N}} \|\iota(x_n)\| = \sup_{n\in\mathbb{N}} \|x_n\| < \infty$$

as desired.

b) Assume first that T is bounded and that $x_n \to x$ weakly. Let $y^* \in Y^*$. Then $y^* \circ T \in X^*$ as a composition of continuous maps and thus

$$y^*(Tx_n) = y^* \circ T(x_n) \to y^* \circ T(x) = y^*(Tx)$$

as claimed.

Now suppose that T is a linear operator which respects weak convergence, but is unbounded. By the second assumption there exists a sequence of unit vectors $(x_n)_n$ in X so that $\|Tx_n\| > n^2$. Define $y_n = \frac{1}{n}x_n$ for every n. Then $(y_n)_n$ converges in the norm topology (and thus also weakly) to 0 because $\|y_n\| = \frac{1}{n}$ tends to 0 as $n \to \infty$. We claim that Ty_n does not converge weakly to 0, giving a contradiction. If Ty_n did converge weakly, then by a) $\sup_{n\in\mathbb{N}} \|Ty_n\|$ would be finite. However $\|Ty_n\| = \frac{1}{n}\|Tx_n\| > n$ for every n.

4. **a)** Assume $(\tilde{a}^n)_n$ is a sequence in $\ell^1(\mathbb{N})$ which converges weakly to a some element $\tilde{a} \in \ell^1(\mathbb{N})$, but not in norm. Define $a^{(n)} = \frac{1}{\|\tilde{a}^n - \tilde{a}\|}(\tilde{a}^n - \tilde{a})$ (passing to a subsequence such that $\|\tilde{a}^n - \tilde{a}\|$ is bounded away from 0). Then $\|a^{(n)}\| = 1$ for all n. Moreover, for all $\phi \in \ell^1(\mathbb{N})^*$ we have by linearity

$$\phi(a^{(n)}) = \frac{1}{\|\tilde{a}^n - \tilde{a}\|}\phi(\tilde{a}^n - \tilde{a}) \to 0$$

since $(\tilde{a}^n)_n$ does not converge in norm.

It thus remains to show that $a_k^{(n)} \to 0$ for $k \in \mathbb{N}$ as $n \to \infty$. To do this, consider the functional $\phi_k \in \ell^1(\mathbb{N})^*$ given by $\phi_k(a) = a_k$ for all $a \in \ell^1(\mathbb{N})$. So the remaining claim follows from weak convergence.

b) First set $K_0 = 1$ and choose n_1 so that $|a_1^{(n_1)}| \leq \frac{1}{5}$. This is possible by the componentwise convergence in part (a). Suppose we have chosen K_{j-1} and n_{j-1} with the desired properties. Choose $n_j > n_{j-1}$ large enough so that $\sum_{k=1}^{K_{j-1}} |a_k^{(n_j)}| \leq \frac{1}{5}$. This is possible by the componentwise convergence to 0 (again) and the fact that we require convergence on only finitely many components. Furthermore we may pick $K_j > K_{j-1}$ so that $\sum_{k=K_j+1}^{\infty} |a_k^{(n_j)}| \leq \frac{1}{5}$ by absolute convergence of the series (that is, $a^{(n_j)} \in \ell^1(\mathbb{N})$). This concludes the construction.

c) Let $b \in \ell^\infty(\mathbb{N})$ as on the exercise sheet. Then for all $j \in \mathbb{N}$

$$\begin{aligned} 1 = \|a^{(n_j)}\|_1 = \sum_{k=1}^{\infty} |a_k^{(n_j)}| &= \sum_{k=1}^{K_{j-1}} |a_k^{(n_j)}| + \sum_{k=K_{j-1}+1}^{I_j} |a_k^{(n_j)}| + \sum_{k=K_j+1}^{\infty} |a_k^{(n_j)}| \\ &= \sum_{k=1}^{K_{j-1}} |a_k^{(n_j)}| + \sum_{i=I_{j-1}+1}^{I_j} a_k^{(n_j)} b_k + \sum_{i=I_j+1}^{\infty} |a_k^{(n_j)}| \end{aligned}$$

where by construction in (b) the first and the last term are $\leq \frac{1}{5}$. Therefore, we have for all $j \in \mathbb{N}$

$$\sum_{k=K_{j-1}+1}^{K_j} a_k^{(n_j)} b_k \geq \frac{3}{5}.$$

From this we may conclude for all j

$$\begin{aligned} \left|\sum_{k=1}^{\infty} a_k^{(n_j)} b_k\right| &= -\sum_{k=1}^{K_{j-1}} |a_k^{(n_j)}| + \sum_{k=K_{j-1}+1}^{I_j} a_k^{(n_j)} b_k - \sum_{k=K_j+1}^{\infty} |a_k^{(n_j)}| \\ &\geq -\frac{1}{5} + \frac{3}{5} - \frac{1}{5} = \frac{1}{5} \end{aligned}$$

where we used the triangle inequality and the fact that $|b_k| = 1$ for all $k \in \mathbb{N}$.

Since $b \in \ell^\infty(\mathbb{N})$, it defines (as usual by dual pairing) a continuous linear functional on $\ell^1(\mathbb{N})$ (see Sheet 8). In particular, as $a^{(n_j)} \to 0$ weakly,

$$\sum_{k=1}^{\infty} a_k^{(n_j)} b_k \to \sum_{k=1}^{\infty} 0 \cdot b_k = 0$$

as $n \to \infty$. This contradicts the inequality

$$\left| \sum_{k=1}^{\infty} a_k^{(n_j)} b_k \right| \geq \frac{1}{5}.$$

5. Let $x \in X$ be fixed. Notice that for any $N \in \mathbb{N}$

$$\tfrac{1}{N} \sum_{n=1}^{N-1} f(T^n x) = \int_X f \, \mathrm{d}\mu_N$$

where we defined the probability measure

$$\mu_N = \tfrac{1}{N} \sum_{n=1}^{N-1} \delta_{T^n x}.$$

We now use the Riesz representation theorem and the Banach-Alaoglu theorem to construct a limit measure of this sequence of measures.

By the Riesz representation theorem (Theorem 7.44) we may identify the set of finite Borel measures on X with the set of positive linear functionals on $C(X)$ via the map

$$\nu \mapsto \Lambda_\nu = (f \in C_c(X) \mapsto \int_X f \, \mathrm{d}\nu) \in C(X)^*.$$

In the literature, it is customary to simply write $\nu(f)$ instead of the notation Λ_ν that we chose here. This identification defines a topology on the set of finite Borel measures on X induced by the weak*-topology on $C(X)^*$. We simply call this the weak*-topology again.

We now restrict this identification to probability measures. Note that for any probability measure ν on X

$$|\Lambda_\nu(f)| \leq \int_X |f| \, \mathrm{d}\nu \leq \|f\|_\infty \nu(X) = \|f\|_\infty$$

for all $f \in C(X)$. So Λ_ν is in the closed unit ball in $C(X)^*$. By the Banach-Alaoglu theorem (Theorem 8.10) the latter unit ball is weak*-compact. In particular, given any sequences of probability measures (ν_n), there is a weak*-converging subsequence $(\nu_{n_j})_j$. If ν is the limit,

$$\int_X f \,\mathrm{d}\nu_n \to \int_X f \,\mathrm{d}\nu$$

for any $f \in C(X)$ by definition of the topology. In particular, ν is also a positive measure (as it defines a positive functional by the convergence) and a probability measure (apply the convergence to $f = 1$).

We now apply this discussion to the measures μ_N and let $(\mu_{N_j})_j$ be a weak*-convergent subsequence with limit $\tilde{\mu}$. We claim that $\tilde{\mu}$ is T-invariant which shows by uniqueness that $\tilde{\mu} = \mu$. This follows from the fact that

$$\int_X (f \circ T - f) \,\mathrm{d}\mu_{N_j} = \tfrac{1}{N_j} \sum_{n=1}^{N_j-1} f(T^{n+1}x) - f(T^n x) = \tfrac{1}{N_j}(f(T^{N_j+1}x) - f(x))$$

goes to zero as $j \to \infty$. Indeed the left hand side converges to $\int_X (f \circ T - f) \,\mathrm{d}\tilde{\mu}$ and the right hand side to 0 as

$$\tfrac{1}{N_j} \|f(T^{N_j+1}x) - f(x)\| \leq \frac{2\|f\|_\infty}{N_j}.$$

Also, the equality

$$\int_X f \circ T \,\mathrm{d}\tilde{\mu} = \int_X f \,\mathrm{d}\tilde{\mu}$$

for all $f \in C(X)$ implies equality for all L^1-functions by density of $C(X)$ in $L^1_{\tilde{\mu}}(X)$. Applying this to characteristic functions yields T-invariance of $\tilde{\mu}$ and thus $\tilde{\mu} = \mu$.

In summary, any converging subsequence of the sequence $(\mu_N)_N$ converges to μ (in the weak*-topology). To upgrade this to convergence of the whole sequence, suppose that $(\mu_N)_N$ does not converge to μ. There exists thus $f \in C(X)$ and $\epsilon > 0$ such that

$$\left| \int_X f \,\mathrm{d}\mu_N - \int_X f \,\mathrm{d}\mu \right| \geq \epsilon$$

holds for N in an infinite set $A \subset \mathbb{N}$. Choosing a subsequence $N_j \in A$ we may again by compactness assume that μ_{N_j} converge, but by the above the limit needs to be μ. Thus, $\mu_N \to \mu$ or in other words

$$\tfrac{1}{N} \sum_{n=1}^{N-1} f(T^n x) = \int_X f \,\mathrm{d}\mu_N \to \int_X f \,\mathrm{d}\mu$$

for every $f \in C(X)$ as desired.

6. **a)** Suppose that U is a weakly open set. By shifting we may assume that $0 \in U$ and pick a neighborhood $N_{\ell_1,\ldots,\ell_n;\epsilon}(0)$ contained in U. In particular,

$$V = \{x \in X : \ell_1(x) = \ldots = \ell_n(x) = 0\} \subset U.$$

If U is bounded, then V (which is a subspace) must be trivial. Now pick $x_1 \in X$ with $\ell_1(x_1) = 1$, $x_2 \in X$ with $\ell_2(x_2) = 1$ and $\ell_2(x_1) = 0$ (if ℓ_2 is not a multiple of ℓ_1) and so forth. Then for any $x \in X$

$$x - \sum_{k=1}^{n} \ell_n(x) x_n \in V$$

and so X is finite-dimensional as V is trivial.

b) The weak closure of the S^1 is the intersection of all weakly closed subsets containing it. By de Morgen's law, the complement is the union of all weakly open sets which contain no element of norm 1. So suppose that x is an element of norm strictly less than 1 which is contained in the complement of the weak closure of S^1, and say it is in a weakly open basis element U. By (a), the open set U contains elements of arbitrarily large norm. Indeed, the argument below shows that U contains elements whose norms are any number between $\|x\|$ and ∞. In particular, U must contain an element of norm 1, a contradiction. Hence the complement of the weak closure of S^1 contains elements of norm no less than 1. Thus, the weak closure of S^1 contains B^1.

Let $x \in X$ with $\|x\| > 1$. We wish to find a weakly open set containing x which misses S^1. By the Hahn-Banach theorem, we know there exists a linear functional on X with $\phi(x) = \|x\| > 1$ and $\|\phi\| = 1$. Then for any $y \in \phi^{-1}((1,\infty))$, we have $1 < \phi(y) \leq \|\phi\|\|y\| = \|y\|$. Hence we have found the desired set.

c) Suppose that $U_1, U_2, \ldots$ is a countable neighborhood basis of $0 \in X$. By replacing these sets with smaller basic open sets, we may assume that

$$U_n = N_{L_n;\epsilon_n}(0)$$

where $L_n \subset X^*$ is finite and $\epsilon_n > 0$. For any $\ell \in X^*$ there must be some $n \in \mathbb{N}$ with $N_{L_n;\epsilon_n}(0) \subset N_{\ell;1}(0)$. Thus, in particular

$$V_n = \{x \in X : \ell(x) = 0 \text{ for all } \ell \in L_n\} \subset N_{\ell;1}(0).$$

More precisely, if $x \in V_n$ then $ax \in V_n$ and so $|\ell(ax)| < 1$ for any scalar a. Thus, $\ell|_{V_n} = 0$ and we may view ℓ as a functional on the quotient space X/V_n. Notice that there is no point in X/V_n on which all functionals induced from L_n vanish identically. Thus, any functional on X/V_n is a linear combination of functionals from L_n and in particular

$$\ell = \sum_{\ell \in L_n} a_\ell \ell$$

for some scalar a_ℓ.

Since ℓ was arbitrary, this shows that any element $\ell \in X^*$ can be expressed as a linear combination of elements in the countable set $\bigcup_{n\in\mathbb{N}} L_n$. By Exercise 2, Sheet 8, this is impossible.