Solutions for exercise sheet 11

1. We only need to check that for any $f \in \mathcal{F}(A)$ non-zero there is a seminorm $\|\cdot\|_a$ which doesn't vanish at f. However, since f is non-zero there is $a \in A$ with $f(a) \neq 0$ and thus $\|f\|_a = |f(a)| > 0$ as desired.

2. We can imitate the proof of Lemma 2.52. Let ℓ be a linear functional on X.

Firstly, assume that ℓ is continuous on X (for the locally convex topology). Then there exists an open neighborhood $N_{\alpha_1,\ldots,\alpha_N;\epsilon}(0)$ of $0 \in X$ such that

$$\ell\big(N_{\alpha_1,\ldots,\alpha_N;\epsilon}(0)\big) \subset B_1(0)$$

If $x \in X$ is arbitrary with $\max_{n=1,\ldots,N}\|x\|_{\alpha_n} \neq 0$ then

$$x' := \frac{\epsilon}{\max_{n=1,\ldots,N}\|x\|_{\alpha_n}} x \in N_{\alpha_1,\ldots,\alpha_N;\epsilon}(0)$$

and so $|\ell(x')| \leq 1$. In other words, using linearity

$$|\ell(x')| = \frac{\epsilon}{\max_{n=1,\ldots,N}\|x\|_{\alpha_n}}|\ell(x)| < 1.$$

This implies $|\ell(x)| \leq \frac{1}{\epsilon}\max_{n=1,\ldots,N}\|x\|_{\alpha_n}$ as desired.

If $x \in X$ satisfies $\max_{n=1,\ldots,N}\|x\|_{\alpha_n} = 0$ then $ax \in N_{\alpha_1,\ldots,\alpha_N;\epsilon}(0)$ for any scalar a. Therefore,

$$|\ell(ax)| = |a||\ell(x)| < 1$$

for any scalar a which must imply that $\ell(x) = 0$. Thus,

$$|\ell(x)| \leq \tfrac{1}{\epsilon}\max_{n=1,\ldots,N}\|x\|_{\alpha_n}$$

for any $x \in X$ as claimed.

For the proof of the converse, note that if

$$|\ell(x)| \leq L \max_{n=1,\ldots,N}\|x\|_{\alpha_n}$$

for some seminorms $\alpha_1, \ldots, \alpha_N$ and all x then

$$\ell\big(N_{\alpha_1,\ldots,\alpha_N;\frac{1}{L+1}}(0)\big) \subset B_1(0).$$

This can be upgraded by translating to continuity of ℓ as in Lemma 2.52.

3. We begin by remarking that the defined topology on $\mathrm{MF}([0,1])$ is also the topology induced by the open neighborhoods

$$U_{\epsilon,\delta}(f_0) = \big\{f \in \mathrm{MF}([0,1]) : \lambda(\{x : |f(x) - f_0(x)| > \epsilon\}) < \delta\big\}.$$

This follows from the fact that $U_{\epsilon,\delta}(f_0) \supset U_{\min\{\epsilon,\delta\}}(f_0)$.

The topology is Hausdorff: if $f_1 \neq f_2$ are elements of $\mathrm{MF}([0,1])$ then there is a positive measure set on which f_1 and f_2 are pointwise distinct. If U_n is the set of points $x \in [0,1]$ with $|f_1(x) - f_2(x)| > \frac{1}{n}$ then $\bigcup_{n\in\mathbb{N}} U_n = U$ and so there must be some U_n which has positive measure. If $f \in \mathrm{MF}([0,1])$ then

$$|f_1(x) - f_2(x)| \leq |f_1(x) - f(x)| + |f(x) - f_2(x)|$$

for x with $|f_1(x) - f(x)| \leq \frac{1}{2n}$ and $|f_2(x) - f(x)| \leq \frac{1}{2n}$ implies that $x \notin U_n$. Another way of saying this is that if $x \in U_n$ then $|f_1(x) - f(x)| > \frac{1}{2n}$ or $|f_1(x) - f(x)| > \frac{1}{2n}$. Let $\delta = \frac{1}{2}\lambda(U_n)$ and $\epsilon = \frac{1}{2n}$. Then the open neighborhoods $U_{\epsilon,\delta}(f_1)$ and $U_{\epsilon,\delta}(f_2)$ are disjoint. Indeed, if there was $f \in U_{\epsilon,\delta}(f_1) \cap U_{\epsilon,\delta}(f_2)$ then

$$\lambda(\{x : |f(x) - f_1(x)| > \tfrac{1}{2n}\}) < \tfrac{1}{2}\lambda(U_n),$$
$$\lambda(\{x : |f(x) - f_2(x)| > \tfrac{1}{2n}\}) < \tfrac{1}{2}\lambda(U_n)$$

and taking the union of the sets appearing on the left gives $\lambda(U_n) < \lambda(U_n)$, which is a contradiction.

Let us prove that addition is continuous. For this, it suffices to show that for any $\epsilon > 0$ and any $f_1, f_2 \in \mathrm{MF}([0,1])$ there exists $\delta > 0$ with

$$U_\delta(f_1) + U_\delta(f_2) \subset U_\epsilon(f_1 + f_2).$$

So let $\delta > 0$ be fixed (to be determined later) and let $g_1 \in U_\delta(f_1)$ and $g_2 \in U_\delta(f_2)$. Then for any $x \in [0,1]$

$$|(f_1 + f_2)(x) - (g_1 + g_2)(x)| \leq |f_1(x) - g_1(x)| + |f_2(x) - g_2(x)|.$$

Now if the left hand side is bigger than ϵ than one of the terms on the right needs to be bigger than $\frac{\epsilon}{2}$. If $\delta < \frac{\epsilon}{2}$ the set of points x which satisfy the latter can have at most measure $\delta + \delta < \epsilon$. This shows that $U_\delta(f_1) + U_\delta(f_2) \subset U_\epsilon(f_1 + f_2)$ whenever $\delta < \frac{\epsilon}{2}$ and in particular continuity of the addition map.

To show continuity of the multiplication, let $\epsilon > 0$, let λ_0 be a scalar and let $f_0 \in \mathrm{MF}([0,1])$. Fix $\delta > 0$ to be determined later. Assume that λ is a scalar with $|\lambda - \lambda_0| < \delta$ and that $f \in U_\delta(f_0)$. Then for any $x \in [0,1]$

$$\begin{aligned} |\lambda_0 f_0(x) - \lambda f(x)| &\leq |\lambda_0||f_0(x) - f(x)| + |\lambda_0 - \lambda||f(x)| \\ &< |\lambda_0||f_0(x) - f(x)| + \delta|f(x)| \\ &\leq (|\lambda_0| + \delta)|f_0(x) - f(x)| + \delta|f_0(x)| \end{aligned}$$

As before, if the left hand side is bigger than ϵ then

$$|f_0(x) - f(x)| > \tfrac{\epsilon}{2(|\lambda_0|+\delta)} \text{ or } |f_0(x)| > \frac{\epsilon}{\delta}$$

Since f_0 is real-valued and measurable, there is a constant $M > 0$ so that the measure of the set of points x with $|f(x)| > M$ is bounded by $\frac{\epsilon}{2}$. Assume that δ is small enough so that $M < \frac{\epsilon}{\delta}$. Also, if δ is small so that $\delta < \frac{\epsilon}{2(|\lambda_0|+\delta)}$ we deduce that $\lambda f \in U_\epsilon(\lambda_0 f_0)$ as desired.

4. **a)** It suffices to show that any neighborhood of 0 of the form $U = N_{\alpha_1,\dots,\alpha_n;\epsilon}(0)$ is convex, balanced and absorbent. By looking at the maximum of the appearing seminorms we may assume that $U = N_{\alpha;\epsilon}(0)$ (see page 293)

For convexity, note that if $x_0, x_1 \in U$ and $\lambda \in [0,1]$ then by the triangle inequality

$$\|\lambda x_0 + (1-\lambda)x_1\|_\alpha \leq \lambda\|x_0\|_\alpha + (1-\lambda)\|x_1\|_\alpha < \lambda\epsilon + (1-\lambda)\epsilon = \epsilon$$

and so $\lambda x_0 + (1-\lambda)x_1 \in U$.

Furthermore, U is balanced as for any $x \in U$ and any scalar λ with $|\lambda| \leq 1$ we have

$$\|\lambda x\|_\alpha = |\lambda|\|x\|_\alpha \leq \|x\|_\alpha < \epsilon.$$

To show that U is absorbent, let $x \in X$ arbitrary and assume that $\|x\|_\alpha \neq 0$ (otherwise $x \in U$ and we would be done). Then

$$\frac{\epsilon}{\|x\|_\alpha} x \in U.$$

This concludes the claim.

b) We need to show that addition and multiplication are continuous. For the former, denote by

$$a : X \times X \to X$$

the addition map and let $U \subset X$ be open. Since it suffices to show continuity on elements of a basis of the topology, we may assume that U is of the form $U = N_{\alpha;\epsilon}(y)$. We need to show that $a^{-1}(U)$ is open. So let $(x_1, x_2) \in a^{-1}(U)$. For any x_1', x_2' with $\|x_1' - x_1\|_\alpha, \|x_2' - x_2\|_\alpha < \delta$ we have

$$\begin{aligned}\|x_1' + x_2' - y\|_\alpha &\leq \|x_1' - x_1\|_\alpha + \|x_1 + x_2 - y\|_\alpha + \|x_2' - x_2\|_\alpha \\ &< 2\delta + \|x_1 + x_2 - y\|\end{aligned}$$

If $\delta > 0$ is chosen small so that $\delta < \frac{1}{2}(\epsilon - \|x_1 + x_2 - y\|)$ then

$$\|x_1' + x_2' - y\|_\alpha < \epsilon$$

which concludes the proof of continuity of the addition map.

For the multiplication map

$$m : X \times \mathbb{K} \to X$$

where $\mathbb{K}$ denotes the ground field we can apply a similar argument. So let $(x, \lambda) \in X \times \mathbb{K}$ and consider the ϵ-neighborhood for some seminorm α. Let (x', λ') be a further point with $\|x' - x\|_\alpha < \delta$ and $|\lambda - \lambda'| < \delta$ for some $\delta \in (0, 1)$. Then

$$\begin{aligned}\|\lambda' x' - \lambda x\|_\alpha \leq |\lambda' - \lambda| \|x'\|_\alpha + |\lambda| \|x' - x\|_\alpha &< \delta(\|x'\|_\alpha + |\lambda|) \\ &< \delta(\|x\|_\alpha + 1 + |\lambda|)\end{aligned}$$

which is $< \epsilon$ if δ is small enough.

5. Let us note that $\mathscr{S}(\mathbb{R}^d)$ is a locally convex vector space as any non-zero function $f \in \mathscr{S}(\mathbb{R}^d)$ satisfies $\|f\|_\infty > 0$. Also, the topology is by definition induced by countably many seminorms as $\mathbb{N}_0^d$ is countable.

It remains to show that $\mathscr{S}(\mathbb{R}^d)$ equipped with the metric

$$\mathrm{d}(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|x^{\varphi_1(n)} \partial_{\varphi_2(n)}(f - g)\|_\infty}{1 + \|x^{\varphi_1(n)} \partial_{\varphi_2(n)}(f - g)\|_\infty}$$

is complete where $\varphi : \mathbb{N} \to (\mathbb{N}_0^d)^2$ is a bijection. One can check that a sequence in $\mathscr{S}(\mathbb{R}^d)$ is a (Cauchy-) sequence if and only if it is a (Cauchy-) sequence with respect to every seminorm – see for instance Exercise 8.66 and Sheet 0. To show that $\mathscr{S}(\mathbb{R}^d)$ is complete, let $(f_n)_n$ be a Cauchy sequence which means by the above that for all α, β

$$\|x^\alpha \partial_\beta (f_n - f_m)\|_\infty \to 0$$

when $n > m \to 0$.

In particular, for any α, β the sequence $(x^\alpha \partial_\beta f_n)_n$ in $C_b(\mathbb{R}^d)$ is a Cauchy sequence. But $C_b(\mathbb{R}^d)$ is a Banach space (see Example 2.24) and thus there is $g_{\alpha,\beta} \in C_b(\mathbb{R}^d)$ with

$$x^\alpha \partial_\beta f_n \to g_{\alpha,\beta}$$

as $n \to \infty$ uniformly. Set $g := g_{0,0} \in C_b(\mathbb{R}^d)$.

We claim that $g_{\alpha,\beta} = x^\alpha \partial_\beta g$ for any α, β (implicitly we also need to show that g is smooth). We begin by proving the case $\alpha = 0$ inductively. Note that for every n and any $h \in \mathbb{R}$

$$f_n(x + he_j) = f_n(x) + \int_0^h \partial_{e_j} f_n(x + te_j)\,\mathrm{d}t.$$

By uniform convergence this implies

$$g(x + he_j) = g(x) + \int_0^h g_{0,e_j}(x + te_j)\,\mathrm{d}t.$$

where the right hand side is differentiable in h with derivative g_{0,e_j}. Applying the above with derivatives of f_n instead of f_n itself one can proceed to show that all partial derivatives of g exist and that $\partial_\beta g = g_{0,\beta}$ for all β. We now only need to show that $g_{\alpha,\beta}(x) = x^\alpha g_{0,\beta}(x)$. This follows from the pointwise (!) convergence of $x^\alpha \partial_\beta f_n$ to either sides.

To conclude we have shown that g is smooth and that $\|x^\alpha \partial_\beta g\|_\infty = \|g_{\alpha,\beta}\|_\infty < \infty$ for all α, β and therefore $g \in \mathscr{S}(\mathbb{R}^d)$. It also follows directly from the construction that g is the limit of the sequence $(f_n)_n$ in the locally convex topology.

6. Let $(f_n)_n$ be a sequence in $C_c(U)$ and let us begin with the simpler claim on the sheet. So assume that there is a compact set $K \subset U$ with $\operatorname{supp}(f_n) \subset K$ for all n and a function $f \in C_c(U)$ with the property $\operatorname{supp}(f) \subset K$ as well as $f_n|_K \to f|_K$ for $n \to \infty$ uniformly. By definition of the topology on $C_c(U)$ from the lecture (see Example 8.63(4)) we need to show that for any arbitrary $F \in C(U)$ we have $f_n F \to fF$ uniformly. But

$$\begin{aligned}\sup_{x \in U} |f_n(x)F(x) - f(x)F(x)| &= \sup_{x \in K} |f_n(x)F(x) - f(x)F(x)| \\ &\leq \|F\|_{K,\infty} \|f_n - f\|_{K,\infty} \to 0\end{aligned}$$

as $n \to \infty$ as desired.

For the converse direction let $f_n \to f \in C_c(U)$ in the topology on $C_c(U)$. Assume by contradiction that there is no compact set $K \subset U$ as above. Since f has compact support, the union of the supports of the f_n's has to be non-compact. We can reword this in the following way: by passing to a subsequence $(f_{n_k})_k$ of $(f_n)_n$ we may find a sequence of points $(x_k)_k$ in U with $f_{n_k}(x_k) \neq 0$ and with $r_k = \mathrm{d}(x_{n_k}, \partial U) \to 0$. By passing to a further subsequence we may also suppose that $r_1 > r_2 > r_3 > \ldots$ and that x_k is not contained in the support of f_ℓ for any $\ell < k$. By applying Urysohn's lemma for every $k \geq 2$ to

$$A_k = \{x : \mathrm{d}(x, \partial U) \leq r_{k+1}\} \cup \bigcup_{\ell < k} \operatorname{supp}(f_\ell)$$

and the point x_k we find a continuous function $g_k \in C_c(U)$ with $g_k(x_k) = 1$ and $g_k|_{A_k} = 0$. Define

$$F = \sum_{k=2}^{\infty} \tfrac{k}{|f_{n_k}(x_k)|} g_k$$

which is an element of $C(U)$ as the sum is finite at every point. For this F we have

$$\|f_{n_k} - f_{n_{k-1}}\|_F \geq \|(f_{n_k} - f_{n_{k-1}})\,|_{A_k}\|_F = \|f_{n_k}|_{A_k}\|_F \geq |f_{n_k}(x_k)||F(x_k)| = k.$$

Thus, the sequence (f_n) cannot converge.