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Solutions for exercise sheet 13

1. We need to show that the graph of $\boldsymbol{\nabla}$

$$\mathcal{G} = \left\{ (f, \boldsymbol{\partial}_1 f, \ldots, \boldsymbol{\partial}_d f) : f \in H^1(\mathbb{T}^d) \right\} \subset H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d)^d$$

is closed. So let $(f_k)_k$ be a sequence in $H^1(\mathbb{T}^d)$ such that

$$f_k \to g \in H^1(\mathbb{T}^d)$$
$$(\boldsymbol{\partial}_1 f_k, \ldots, \boldsymbol{\partial}_d f_k) \to (g_1, \ldots, g_k) \in L^2(\mathbb{T}^d)^d$$

as $k \to \infty$. Recall that we interpret $H^1(\mathbb{T}^d)$ as a subspace of $L^2(\mathbb{T}^d)$ (as a function determines its weak derivatives) but in fact defined the Sobolov space $H^1(\mathbb{T}^d)$ as a closure in $L^2(\mathbb{T}^d)^d$. More precisely, g (as well as f_k) as an element of $H^1(\mathbb{T}^d)$ should be viewed as a tuple

$$(g, \boldsymbol{\partial}_1 g, \ldots, \boldsymbol{\partial}_d g)$$

where by Lemma 5.2, $\boldsymbol{\partial}_1 g, \ldots, \boldsymbol{\partial}_d g$ are uniquely determined by By definition of the Sobolev space, $f_k \to g \in H^1(\mathbb{T}^d)$ means that

$$(f_k, \boldsymbol{\partial}_1 f_k, \ldots, \boldsymbol{\partial}_d f_k) \to (g, \boldsymbol{\partial}_1 g, \ldots, \boldsymbol{\partial}_d g)$$

as $k \to \infty$ in L^2. This shows that $\boldsymbol{\partial}_1 g = g_1, \ldots, \boldsymbol{\partial}_d g = g_d$ and hence

$$(g, g_1, \ldots, g_d) = (g, \boldsymbol{\partial}_1 g, \ldots, \boldsymbol{\partial}_d g) \in \mathcal{G}$$

as claimed.

2. Let us first note that the formula

$$\int_{\mathbb{T}^d} \psi(x) \partial_\alpha f(x) \, \mathrm{d}x = (-1)^{\|\alpha\|_1} \int_{\mathbb{T}^d} \partial_\alpha \psi(x) f(x) \, \mathrm{d}x \tag{1}$$

holds by partial integration for any $f \in C^\infty(\mathbb{T}^d)$ and $\psi \in C^\infty(\mathbb{T}^d)$.

- Assume that $f \in H^k(\mathbb{T}^d)$ and denote by $(f_\alpha)_{\|\alpha\|_1 \le k}$ the corresponding tuple of functions in $L^2(\mathbb{T}^d)$. By definition, there exists a sequence of functions $g_k \in C^\infty(\mathbb{T}^d)$ so that $\partial_\alpha g_k \to f_\alpha$ in L^2 as $k \to \infty$ for any α. Now note that the left-hand side of (1) when applied to g_k must converge to
$$\int_{\mathbb{T}^d} \psi(x) f_\alpha(x) \,\mathrm{d}x.$$
This is in essence a consequence of the Cauchy-Schwarz inequality. Applying the same reasoning to the right-hand side we obtain
$$\int_{\mathbb{T}^d} \psi(x) f_\alpha(x) \,\mathrm{d}x = (-1)^{\|\alpha\|_1} \int_{\mathbb{T}^d} \partial_\alpha \psi(x) f(x) \,\mathrm{d}x$$
for any $\psi \in C^\infty(\mathbb{T}^d)$. Since $f_\alpha \in L^2(\mathbb{T}^d)$ we have thus found the α-weak derivative of f.
- Now suppose that all α-weak derivatives for $\|\alpha\|_1 \le k$ exist and denote them by f_α. Let us compute the Fourier series of f_α: Applying the definition of the weak derivative to $\psi = \chi_{-n}$ for $n \in \mathbb{Z}^d$ we obtain that
$$a_n(f_\alpha) = \int_{\mathbb{T}^d} \chi_{-n}(x) f_\alpha(x) \,\mathrm{d}x = (-1)^{\|\alpha\|_1} \int_{\mathbb{T}^d} \partial_\alpha \chi_{-n}(x) f(x) \,\mathrm{d}x.$$
Now note that
$$\begin{aligned} \partial_\alpha \chi_{-n}(x) &= (2\pi \mathrm{i})^{\|\alpha\|_1} (-n_1)^{\alpha_1} \cdots (-n_d)^{\alpha_d} \chi_{-n}(x) \\ &= (2\pi \mathrm{i})^{\|\alpha\|_1} (-1)^{\|\alpha\|_1} n^\alpha \chi_{-n}(x) \end{aligned}$$
and so
$$a_n(f_\alpha) = (2\pi \mathrm{i})^{\|\alpha\|_1} n^\alpha a_n(f).$$
Let $g_N = \sum_{n=-N}^{N} a_n(f) \chi_n$. Since $f_\alpha \in L^2(\mathbb{T}^d)$ and the Fourier series of f_α has the above shape, $\partial_\alpha g_N \to f_\alpha$ in L^2 for any α with $\|\alpha\|_1 \le k$. This shows that $f \in H^k(\mathbb{T}^d)$ as desired.

3. To fix some notation, let us denote by x_i, $0 \le i \le n$, the points where the derivative f' does not exist. To simplify notational matters a bit, let us also assume that $x_0 = 0$ is one of these points (this is just throwing away information). We view f and all other functions on $\mathbb{T}$ below as functions on $[0, 1]$ (with periodicity of course).

a) Let $\psi \in C^\infty(\mathbb{T})$ and fix a closed interval $[a, b] \subset (x_{i-1}, x_i)$ for some $i \ge 1$. Then by the fundamental theorem of calculus
$$\begin{aligned} \int_a^b \psi(x) f'(x) \,\mathrm{d}x &= [\psi(x) f(x)]_a^b - \int_a^b \psi'(x) f(x) \,\mathrm{d}x \\ &= \psi(b) f(b) - \psi(a) f(a) - \int_a^b \psi'(x) f(x) \,\mathrm{d}x. \end{aligned}$$

Since f is continuous (also at the points x_i) the limit of the right-hand side for $a \to x_{i-1}$ and $b \to x_i$ exists and so

$$\int_{x_{i-1}}^{x_i} \psi(x) f'(x) \, \mathrm{d}x = \psi(x_i) f(x_i) - \psi(x_{i-1}) f(x_{i-1}) - \int_{x_{i-1}}^{x_i} \psi'(x) f(x) \, \mathrm{d}x.$$

(The left-hand side can be viewed as a Lebesgue integral or a improper Riemann integral.) Summing over i we therefore obtain

$$\begin{aligned}\int_0^1 \psi(x) f'(x) \, \mathrm{d}x &= \sum_{i=1}^n \psi(x_i) f(x_i) - \psi(x_{i-1}) f(x_{i-1}) - \int_0^1 \psi'(x) f(x) \, \mathrm{d}x \\ &= \psi(1) f(1) - \psi(0) f(0) - \int_0^1 \psi'(x) f(x) \, \mathrm{d}x \\ &= - \int_0^1 \psi'(x) f(x) \, \mathrm{d}x\end{aligned}$$

using periodicity. This shows the claim in a).

b) By a) and Exercise 2 it suffices to show that $f' \in L^2(\mathbb{T})$. For this, it suffices to show that $f'|_{[x_{i-1}, x_i]}$ for $i \geq 1$ is in L^2. However, this follows directly from the fact that $f'|_{[x_{i-1}, x_i]}$ is continuous as the the one-sided limits at the endpoints are assumed to exist.

4. Assume first that $f \in H^1(\mathbb{T})$. For the converse we will apply the same strategy as in Exercise 3. As $f \in H^1(\mathbb{T})$ there exists a weak derivative $g \in L^2(\mathbb{T})$ as in Exercise 2. We first claim that $g(x) = f'(x)$ almost everywhere. Here, the derivative of f is defined i.e. at points $x \neq 0$. Let $\psi \in C^\infty(\mathbb{T})$ be such that $\operatorname{supp}(\psi) \subset (0, 1)$. Then by partial integration

$$\int_{\mathbb{T}} \psi(x) f'(x) \, \mathrm{d}x = \psi(1) - \psi(0) - \int_{\mathbb{T}} \psi'(x) f(x) \, \mathrm{d}x = - \int_{\mathbb{T}} \psi'(x) f(x) \, \mathrm{d}x$$

where the left-hand side makes sense by the assumption on the support of ψ. Combining this with the definition of the weak derivative we see that

$$\int_{\mathbb{T}} \psi(x) (f'(x) - g(x)) \, \mathrm{d}x = 0$$

Since the function ψ as above are dense in $L^2(\mathbb{T})$, this shows that $f' = g$ almost everywhere. In explicit formulas,

$$f'(x) = \varkappa x^{\varkappa - 1}$$

for $x \in (0,\delta)$. Therefore,

$$\infty > \|f'\|_{L^2}^2 \geq \varkappa^2 \int_0^\delta x^{2\varkappa-2}\,\mathrm{d}x.$$

and so the latter integral exists. By explicit integration, this is the case if and only if $2\varkappa - 2 > -1$ i.e. if and only if $\varkappa > \frac{1}{2}$.

Now conversely assume that $\varkappa > \frac{1}{2}$. Then f' defines an L^2-function (essentially by the argument just given) as it is in L^2 on each of the intervals $[0,\delta]$, $[\delta, 1-\delta]$ and $[1-\delta, 1]$. It remains to show that f' is indeed the weak derivative. As before, one immediately checks by partial integration that

$$\int_{\mathbb{T}} \psi(x) f'(x)\,\mathrm{d}x = -\int_{\mathbb{T}} \psi'(x) f(x)\,\mathrm{d}x$$

for all $\psi \in C^\infty(\mathbb{T})$ with $\mathrm{supp}(\psi) \subset (0,1)$. It remains to settle the case where $\psi(0) \neq 0$. By an Urysohn-type of argument (or rather a partition of unity) we may assume that $\mathrm{supp}(\psi) \cap [\delta, 1-\delta] = \emptyset$. Therefore (viewing ψ and f as functions on $(-\delta,\delta)$),

$$\begin{aligned}\int_{-\delta}^{\delta} \psi(x) f'(x)\,\mathrm{d}x &= \int_0^\delta \psi(x) f'(x)\,\mathrm{d}x = \lim_{a\searrow 0}\int_a^\delta \psi(x) f'(x)\,\mathrm{d}x \\ &= -\lim_{a\searrow 0}\int_a^\delta \psi'(x) f(x)\,\mathrm{d}x = -\int_0^\delta \psi'(x) f(x)\,\mathrm{d}x \\ &= -\int_{-\delta}^{\delta} \psi'(x) f(x)\,\mathrm{d}x\end{aligned}$$

where we used the fact that f vanishes identically on the left of 0 and where the first limit exists as the second one exists. This concludes the exercise.

5. **a)** Let $x, y \in [0,1)$ and assume without loss of generality that $x < y$ and that the absolute value $|y-x|$ is indeed the distance (otherwise one can work in the interval $[-\frac{1}{2}, \frac{1}{2}]$. We estimate using the fundamental theorem of calculus

$$|f(y) - f(x)| \leq \int_x^y |f'(t)|\,\mathrm{d}t = \int_{\mathbb{T}} \mathbb{1}_{[x,y]}(t)|f'(t)|\,\mathrm{d}t \leq \|\mathbb{1}_{[x,y]}\|_{L^2}\|f'\|_{L^2}$$

by the Cauchy-Schwarz inequality. Now note that

$$\|\mathbb{1}_{[x,y]}\|_{L^2}^2 = \int_x^y \mathrm{d}t = y - x = |y-x|.$$

b) By part a) we have

$$\sup_{x,y\in\mathbb{T}} \frac{|f(y)-f(x)|}{|y-x|^{\frac{1}{2}}} \leq \|f'\|_{L^2} \leq \|f\|_{H^1}$$

for all $f \in C^\infty(\mathbb{T})$. Also, by the Sobolev embeddings theorem

$$\|f\|_\infty \ll \sqrt{\|f\|_{L^2}^2 + \|f'\|_{L^2}}$$

Therefore, the inclusion operator $C^\infty(\mathbb{T}) \to C^{0,\frac{1}{2}}(\mathbb{T})$ is bounded when $C^\infty(\mathbb{T})$ is equipped with the $H^1(\mathbb{T})$-norm. Since $C^\infty(\mathbb{T}) \subset H^1(\mathbb{T})$ is dense, the inclusion operator extends uniquely to the completion $H^1(\mathbb{T})$.

c) The argument here is exactly the same as in the proof of Theorem 5.6.

6. **a)** Notice that the closure of the image of the unit ball in $C(\mathbb{T}^d)$ contains the characters χ_n for $n \in \mathbb{Z}^d$. The set of characters certainly does not have compact closure as by orthogonality for any $m \neq n$

$$\|\chi_n - \chi_m\|_{L^2} = \sqrt{2}.$$

b) Let $U \subset \mathbb{R}^d$ be bounded and open. Let $f_n \in C_b^{k+1}(U) \subset C_b^k(U)$ for every $n \in \mathbb{N}$ define a sequence with $\|f_n\|_{C_b^{k+1}(U)} \leq 1$. The norm we consider on the space $C_b^k(U)$ here is given by

$$\|f\|_{C_b^k(U)} = \max_{\|\alpha\|_1 \leq k} \|\partial_\alpha f\|_\infty$$

for $f \in C_b^k(U)$. Note that this is just a convenient choice and any other, similarly defined, norm would also do.

Fix some $\alpha \in \mathbb{N}_0^d$ with $\|\alpha\|_1 \leq k$. We claim that the sequence $(\partial_\alpha f_n)_n$ contains a Cauchy-sequence in $C_b(U)$ and would like to invoke the theorem of Arzela-Ascoli for this. Though this is not the kind of space we may Arzela-Ascoli as formulated in Theorem 2.38 to (that version applies to $C(X)$ where X is a compact metric space) the proof applies verbatim – see also Sheet 3. We need to show that $\{\partial_\alpha f_n : n \in \mathbb{N}\}$ is equicontinuous at every point. So let $x \in U$ and $y \in U$ be such that the straight line from x to y is contained in U. Then by the fundamental theorem of calculus

$$\partial_\alpha f(y) - \partial_\alpha f(y) = \int_0^1 \sum_{i=1}^d \partial_i \partial_\alpha f((1-t)x + ty)(x_i - y_i)\, \mathrm{d}t$$

and so

$$|\partial_\alpha f(y) - \partial_\alpha f(y)| \ll \|f\|_{C_b^{k+1}(U)} \|x - y\|$$

where the implicit constant is absolute (i.e. does not depend on anything). This shows equicontinuity and thus $(\partial_\alpha f_n)_n$ contains a Cauchy-sequence in $C_b(U)$.

Since we are only considering finitely many α's and α above was arbitrary we may find a subsequence of $(f_n)_n$ in $C_b^k(U)$ which is Cauchy. This proves the claim.

It remains to show that claim about $U = \mathbb{R}$. Let $f \in C_c^\infty(\mathbb{R})$ be non-trivial with compact support in $(0, 1)$ and so that $\|f\|_{C_b^{k+1}(\mathbb{R}))} \leq 1$. Then set for any $\ell \in \mathbb{N}$

$$f_\ell : x \in \mathbb{R} \mapsto f(x - \ell).$$

Note that the support of f_ℓ is contained in $(\ell, \ell + 1)$ by definition. We may thus conclude that

$$\|f_{\ell_1} - f_{\ell_2}\|_\infty = \|f\|_\infty > 0.$$

Thus, no subsequence of $(f_\ell)_\ell$ can be a Cauchy-sequence in $C_b^k(\mathbb{R})$. This shows the remaining claim.

c) Define K to be the closure of the image of the unit ball in $H^1(\mathbb{T})$ inside $L^2(\mathbb{T})$. Since $L^2(\mathbb{T})$ is complete and K is closed, K is complete. We shall show that K is totally bounded and begin by proving that elements of K have uniformly small tails. Let $f \in K$ be in the image of the unit ball of $H^1(\mathbb{T})$ and let f' be the weak derivative of f. Then by Lemma 5.2,

$$\sum_{n \in \mathbb{Z}} n^2 |a_n(f)|^2 \ll \|f'\|_{L^2} \leq 1.$$

where $a_n(f)$ denotes the n-th Fourier coefficient. For any $N \in \mathbb{N}$ we then have

$$\sum_{|n| \geq N} |a_n(f)|^2 = \tfrac{1}{N^2} \sum_{|n| \geq N} N^2 |a_n(f)|^2 \leq \tfrac{1}{N^2} \sum_{|n| \geq N} n^2 |a_n(f)|^2 \ll \frac{1}{N^2}$$

where the implicit constant is the same as above and in particular does not depend on N. By continuity the inequality also holds for any other $f \in K$.

We now prove that K is totally bounded. So let $\epsilon > 0$ and let $N \in \mathbb{N}$ be small enough so that

$$\sum_{|n| \geq N} |a_n(f)|^2 \leq \tfrac{\epsilon^2}{4}$$

Denote by

$$K' = \left\{ f \in K : f = \sum_{|n| \leq N} a_n(f) \chi_n \right\}$$

which is a closed, bounded subset of a finite-dimensional space and thus compact. We may hence choose a finite subset $A \subset K'$ so that for any $f \in K'$ there is $a \in A$ with $\|f - a\|_2 < \frac{\epsilon}{2}$.

Let $f \in K$ be arbitrary, set $\tilde{f} = \sum_{|n| \leq N} a_n(f)\chi_n$ and choose $a \in A$ with the property $\|\tilde{f} - a\|_2 < \frac{\epsilon}{2}$. Then

$$\|f - a\|_2 \leq \|f - \tilde{f}\|_2 + \|\tilde{f} - a\|_2 < \left(\sum_{|n| \geq N} |a_n(f)|^2 \right)^{\frac{1}{2}} + \tfrac{\epsilon}{2} \leq \epsilon$$

In other words, the balls of radius $\epsilon > 0$ around the points in the finite set A cover K. Since $\epsilon > 0$ was arbitrary, K is totally bounded.

7. **a)** As noted in the hint, we only need to show that a linear combination of compact operators is compact – see Lemma 6.3. So let λ be a scalar and let $L_1, L_2 \in B(X)$ be compact. Then

$$\begin{aligned}\overline{(\lambda L_1 + L_2)(B_1^X)} &\subset \overline{\lambda L_1(B_1^X) + L_2(B_1^X)} \subset \overline{\lambda L_1(B_1^X)} + \overline{L_2(B_1^X)} \\ &= \lambda \overline{L_1(B_1^X)} + \overline{L_2(B_1^X)}\end{aligned}$$

by continuity of addition and multiplication. By the same continuity, the latter set is compact and so $\overline{(\lambda L_1 + L_2)(B_1^X)}$ is compact as desired.

b) It follows directly from Lemma 6.7 that $K(X)$ is a closed ideal. In particular, the Calkin algebra, when equipped with the quotient norm, forms a Banach space. It remains to show that the quotient norm satisfies the required property of a Banach algebra. For this, let $A_1, A_2 \in B(X)$ and given $\epsilon > 0$ choose $K_1, K_2 \in K(X)$ with

$$\|A_1 + K_1\| \leq \|A_1 + K(X)\| + \epsilon, \quad \|A_2 + K_2\| \leq \|A_2 + K(X)\| + \epsilon.$$

To simplify notation set $A_1' = A_1 + K_1$ and $A_2' = A_2 + K_2$. Then by definition of the quotient and the norm $\|A_1' + K(X)\| = \|A_1 + K(X)\|$ and similarly for A_2'. Also, $A_1 A_2 + K(X) = A_1' A_2' + K(X)$. Then

$$\begin{aligned}\|A_1 A_2 + K(X)\| &\leq \|A_1' A_2'\| \leq \|A_1'\| \|A_2'\| \\ &\leq \|A_1 + K(X)\| \|A_2 + K(X)\| \\ &\quad + \big(\|A_1 + K(X)\| + \|A_2 + K(X)\| + \epsilon\big)\epsilon\end{aligned}$$

and as $\epsilon > 0$ was arbitrary this shows

$$\|A_1 A_2 + K(X)\| \leq \|A_1 + K(X)\| \|A_2 + K(X)\|$$

as desired.

8. **a)** As proved in the lecture, operators with finite-dimensional range are compact. Now it suffices to note that all P_n's are of this shape.

b) The definition of the strong operator topology is such that $P_n \to \mathrm{id}$ if and only if $P_n(x) \to x$ for all $x \in \ell^2(\mathbb{N})$ as $n \to \infty$. See also Exercise 8.57. But now note that for any $x \in \ell^2(\mathbb{N})$

$$\|x - P_n(x)\|_{\ell^2}^2 = \sum_{k=n+1}^{\infty} |x_k|^2 \to 0$$

as $n \to \infty$ by absolute convergence. Thus, $P_n \to \mathrm{id}$ in the strong operator topology.

The identity operator on any infinite-dimensional Banach space X is not compact. Indeed, $\overline{\mathrm{id}(B_1^X)} = \overline{B_1^X}$ is not compact by Proposition 2.35.

9. For simplicity of the exposition, let us first assume that $T = \mathrm{id}_X - K$ is injective. We thus have a bijective bounded operator $T : X \to T(X)$ and so in order to prove that $T(X)$ is closed we may also prove that T^{-1} is bounded. In other words, if there is a constant $C > 0$ such that

$$\|x\| \leq C\|Tx\| \tag{2}$$

for all $x \in X$ then any Cauchy-sequence $Tx_n = x_n - Kx_n$ in $T(X)$ has the property that x_n is a Cauchy-sequence in X. Since X is a Banach space, the limit x exists. It satisfies $Tx_n \to Tx$ then as required showing that $T(X)$ is complete and hence closed. So suppose that such a constant C does not exist and choose $(x_n)_n$ in X so that

$$\|x_n\| = 1 \text{ and } 1 > n\|Tx_n\|.$$

Thus, $Tx_n = x_n - Kx_n \to 0$. But $Kx_n \in K(B_1^X)$ where $K(B_1^X)$ has compact closure by choice of K. Thus, there is $y \in X$ so that $Kx_n \to y$. This implies that

$$x_n = (x_n - Kx_n) + Kx_n \to 0 + y = y.$$

Thus, $Kx_n \to Ky$ as K is bounded. But since $Kx_n \to y$ also holds, we have $Ky = y$ which implies $y = 0$ by assumption. We also know that $\|x_n\| = 1$ for all n which is impossible as $x_n \to y = 0$. This is a contradiction and hence the inequality (2) holds for some constant C. We already proved that this implies the claim.

We now drop the assumption that $T = \mathrm{id}_X - K$ is injective and let V be the kernel of $\mathrm{id}_X - K$. Notice that V is finite-dimensional. Indeed, $K|_V : V \to X$ is compact and equal to the identity operator, which can only be compact if V is finite-dimensional by Proposition 2.35. We know from Exercise 6, Sheet 9 that V has a closed complement. Denote this complement by W. Notice that the image of $\mathrm{id}_X - K$ restricted to W is equal to the image of the whole space X under the same map as V is the kernel. Furthermore, W is a Banach space and $K|_W$ is compact. One can therefore apply the argument from the first part of the exercise to the new injective operator $(\mathrm{id}_X - K)|_W$ and obtain the claim in the general case.

10. **a)** Recall that the adjoint operator of, say, $L \in B(\mathcal{H})$ is characterized uniquely by the equation

$$\langle Lv, w \rangle = \langle v, L^* w \rangle$$

for all $v, w \in \mathcal{H}$.

Now consider the operator $L = aS + bT$. We see that for any $v, w \in \mathcal{H}$

$$\begin{aligned}\langle (aS + bT)v, w \rangle &= \langle aSv + bTv, w \rangle = a \langle Sv, w \rangle + b \langle Tv, w \rangle \\ &= a \langle v, S^* w \rangle + b \langle v, T^* w \rangle = \langle v, \overline{a} S^* w \rangle + \left\langle v, \overline{b} T^* w \right\rangle \\ &= \left\langle v, (\overline{a} S^* + \overline{b} T^*) w \right\rangle\end{aligned}$$

and so the claim follows from the above uniqueness statement.

Similarly, we compute for $L = ST$ and $v, w \in \mathcal{H}$

$$\langle STv, w \rangle = \langle Tv, S^* w \rangle = \langle v, T^* S^* w \rangle$$

which concludes part a).

b) We first prove the equality $\operatorname{im}(T)^\perp = \ker(T^*)$. So let $v \in \ker(T^*)$ and let $w \in \mathcal{H}$. Then $\langle v, Tw \rangle = \langle T^* v, w \rangle = 0$ implies that $v \in \operatorname{im}(T)^\perp$. Conversely, if $v \in \operatorname{im}(T)^\perp$ then $\langle T^* v, w \rangle = \langle v, Tw \rangle = 0$ for all $w \in \mathcal{H}$ implies that $T^* v = 0$.

We now prove the equality $\ker(T)^\perp = \overline{\operatorname{im}(T^*)}$ or by taking orthogonal complements rather the equivalent statement

$$\ker(T) = \operatorname{im}(T^*)^\perp$$

Replacing T with T^* and using that $(T^*)^* = T$, this is in fact the same equality as the one we already proved.

c) Assume that T is unitary i.e. that T is surjective and that

$$\|Tv\|^2 = \langle Tv, Tv \rangle = \|v\|^2.$$

for all $v \in \mathcal{H}$ We first claim that $\langle Tv, Tw \rangle = \langle v, w \rangle$ for all $v, w \in \mathcal{H}$. By the polarization identity (see Exercise 3, Sheet 4)

$$\begin{aligned}\langle Tv, Tw \rangle &= \tfrac{1}{4} \sum_{k=0}^{4} \mathrm{i}^k \|Tv + \mathrm{i}^k Tw\| = \tfrac{1}{4} \sum_{k=0}^{4} \mathrm{i}^k \|T(v + \mathrm{i}^k w)\| \\ &= \tfrac{1}{4} \sum_{k=0}^{4} \mathrm{i}^k \|v + \mathrm{i}^k w\| = \langle v, w \rangle\end{aligned}$$

and so this proves the claim. As in a), the equality

$$\langle v, T^*Tw\rangle = \langle Tv, Tw\rangle = \langle v, w\rangle$$

proves that $T^*T = \mathrm{id}_{\mathcal{H}}$. Since this shows that $T^{-1} = T^*$ as T is surjective, we also have $TT^* = \mathrm{id}_{\mathcal{H}}$.

Let us turn to the converse and assume that $T^*T = TT^* = \mathrm{id}_{\mathcal{H}}$. By the latter equality, T must be surjective. Furthermore, we have for all $v \in \mathcal{H}$

$$\langle v, v\rangle = \langle v, T^*Tv\rangle = \langle Tv, Tv\rangle$$

which finishes the exercise.

11. We do not give the statement right away, but rather discover it while using all the theory we can. Let us apply Theorem 6.27 to the self-adjoint compact operator T_2. It follows there is a sequence λ_n of non-zero distinct eigenvalues with $\lambda_n \to 0$ so that we can write

$$\mathcal{H} = \mathcal{H}_0 \oplus \bigoplus_n \mathcal{H}_n \tag{3}$$

where $\mathcal{H}_0$ is the kernel of T_2 and $\mathcal{H}_n$ is the eigenspace of T_2 to the eigenvalue λ_n. By assumption on T_2, the kernel $\mathcal{H}_0$ is trivial. Also, it is contained in Theorem 6.27 that each eigenspace $\mathcal{H}_n$ is finite-dimensional.

We claim that the eigenspaces $\mathcal{H}_n$ are also invariant under T_1. Indeed, if $v \in \mathcal{H}_n$ then by commutativity

$$T_2(T_1v) = T_1T_2v = \lambda_nT_1v$$

and so $T_1v \in \mathcal{H}_n$.

Recall that any self-adjoint operator on a finite-dimensional inner product space is diagonalizable! Applying this for $n \in \mathbb{N}$ to $T_1|_{\mathcal{H}_n}$ we find an orthonormal basis $\mathcal{B}_n$ of $\mathcal{H}_n$ of eigenvectors of T_1. Since $T_2|_{\mathcal{H}_n}$ is a scalar multiplication map, this orthonormal basis also consist of eigenvectors of T_2. Let $v_1, v_2, \ldots, v_k, \ldots$ be an enumeration of $\bigcup_n \mathcal{B}_n$. By Equation 3 this forms an orthonormal basis of $\mathcal{H}$ of eigenvectors for T_1, T_2 simultaneously. By the spectral theorem as applied above, we have that the eigenvalue of v_k goes to zero as k goes to infinity.

12. Recall from the discussion after the proof of Lemma 2.68 that the functions

$$s_n : x \in [0, 1] \mapsto \sin(\pi nx)$$

for $n \in \mathbb{N}$ satisfy $K(s_n) = \frac{-1}{(\pi n)^2} s_n$ and are therefore eigenvectors of the Hilbert-Schmidt operator K with eigenvalues $\mu_n = \frac{-1}{(\pi n)^2}$. It remains to show that these are all eigenvalues and that all these eigenvalues have (geometric) multiplicity one.

To achieve this, suppose that f is an eigenvector of K with eigenvalue $\lambda \neq 0$. In particular, $Kf = \lambda f$. By Lemma 2.68 this is another way of saying that $f \in C^2([0,1])$ with $\lambda f'' = f$ and $f(0) = f(1) = 0$. Thus, since $f'' = \frac{1}{\lambda} f \in C^2([0,1])$ we have that $f \in C^4([0,1])$ and continuing in this fashion, f must be smooth.

Suppose that $\lambda < 0$. By uniqueness of solutions to second order ODE's we must have

$$f(x) = A \sin(\sqrt{|\lambda|}x) + B \cos(\sqrt{|\lambda|}x)$$

for all $x \in [0,1]$. Since $f(1) = 0$, $B = 0$. Since $f(1) = 0$, $\sqrt{|\lambda|} \in \pi\mathbb{Z}$ and so $\sqrt{|\lambda|} = \pi n$ for some $n \in \mathbb{N}$. Thus, $f(x) = A \sin(\pi n x)$ as desired.

Suppose that $\lambda > 0$. Then

$$f(x) = A \sinh(\sqrt{\lambda}x) + B \cosh(\sqrt{\lambda}x)$$

for all x and as $f(0) = 0$ we have $B = 0$. Since $\sinh' = \cosh$ is positive, $\sinh$ is strictly increasing and therefore $f(1) > f(0) = 0$ so there is no eigenvector to positive eigenvalue.

Remark: An alternative approach uses Fourier series on the interval. For a smooth function (such as an eigenfunction) the Fourier series is absolutely convergent. If an eigenfunction apart linearly independent from the s_n would exist, it would need to be orthogonal to these. But these functions form a orthogonal basis of L^2 (see Exercise 3.55).

It remains to compute the kernel of K. So suppose that $Kf = 0$. The proof of Lemma 2.68 then shows that (taking the second derivative of Kf) $f = 0$ so the kernel is trivial.