

Probability Theory

Exercise sheet 1

Exercise 1.1 Consider the Probability space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \mathbb{P})$ where $\mathbb{P}(dx) = \frac{1}{2\pi} \exp\{-\frac{1}{2}(x_1^2 + x_2^2)\} dx$ with $x = (x_1, x_2)$ for $x \in \mathbb{R}^2$ and dx the Lebesgue measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$. Find the distribution of the random variable

$$Z : x = (x_1, x_2) \in \mathbb{R}^2 \mapsto x_1^2 + x_2^2 \in \mathbb{R}.$$

Exercise 1.2 Let $\mathcal{Z} := (A_i)_{i \in I}$ be a countable decomposition of a set $\Omega \neq \emptyset$ in “atoms” A_i , that is $\Omega = \bigcup_{i \in I} A_i$, where $A_i \cap A_k = \emptyset$ for $i \neq k$, and I countable.

- (a) Show that the σ -algebra generated by \mathcal{Z} is of the form

$$\sigma(\mathcal{Z}) = \left\{ \bigcup_{i \in J} A_i \mid J \subseteq I \right\}.$$

Hint: Recall the definition of $\sigma(\mathcal{Z})$.

- (b) Show that the family of $\sigma(\mathcal{Z})$ -measurable random variables is exactly the family of functions on Ω that are constant on “atoms” (that is, all functions f such that for each i , f is constant on A_i).

Exercise 1.3 Let Ω be a non-empty set and let $X : \Omega \rightarrow \mathbb{R}$ and $Y : \Omega \rightarrow \mathbb{R}$ be two functions. The σ -algebra on Ω generated by X is defined by $\sigma(X) := \{X^{-1}(B) \mid B \in \mathcal{R}\}$, where \mathcal{R} denotes the Borel σ -algebra on \mathbb{R} . In this exercise we will show that:

Claim: Y is $\sigma(X)$ - \mathcal{R} -measurable \iff there exists an \mathcal{R} - \mathcal{R} -measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$, such that $Y = f \circ X$.

Hint: For (b)–(e), cf. the proof of (1.2.16) in the lecture notes.

- (a) Show the \Leftarrow direction.
- (b) Show the \implies direction for any Y of the form $Y = 1_A$, where $A \in \sigma(X)$.
- (c) Show the \implies direction for any Y that is a linear combination of indicator functions, i.e. for Y of the form $Y = \sum_{i=1}^n c_i 1_{A_i}$, where $n \in \mathbb{N}$, $c_1, \dots, c_n \in \mathbb{R}$ and $A_1, \dots, A_n \in \sigma(X)$.
- (d) Show the \implies direction for any Y such that $Y \geq 0$.
- (e) Complete the proof of the claim (i.e. show the \implies direction for an arbitrary Y).

Submission deadline: 13:15, Oct 2

Location: During exercise class or in the tray outside of HG G53–54.

Class assignment:

Students	Time & Date	Room	Assistant
An-Gr	Tue 13-14	HG F 26.5	Daniel Balint
He-Lang	Tue 13-14	ML H 41.1	Daniel Contreras Salinas
Lanz-Sa	Tue 14-15	HG F 26.5	Daniel Balint
Sch-Zh	Tue 14-15	ML H 41.1	Chong Liu

Office hours (Präsenz); Mon. and Thu., 12:00 - 13:00 in HG G32.6.

Exercise sheets and further information are also available on:

<http://metaphor.ethz.ch/x/2018/hs/401-3601-00L/>