

# Probability Theory

## Exercise sheet 3

**Exercise 3.1** Assume that  $X_k = \frac{1}{k^2} + \frac{Z_k}{k^{\frac{3}{4}}}$ , for  $k \geq 1$ , where  $Z_k$  are i.i.d random variables with  $P[Z_k = 1] = P[Z_k = -1] = \frac{1}{4}$  and  $P[Z_k = 0] = \frac{1}{2}$ . Discuss the convergence of the random series  $\sum_{k \geq 1} X_k$ .

**Exercise 3.2** Let  $\mathcal{M}$  be the set of the real-valued random variables on the probability space  $(\Omega, \mathcal{A}, P)$ . We define on  $\mathcal{M}$  an equivalence relation as follows:

$$X \sim Y \quad :\iff \quad P(X = Y) = 1$$

We denote by  $\mathcal{M}/\sim$  the set of equivalence classes in  $\mathcal{M}$  with respect to  $\sim$  and we denote by  $[X]$  the equivalence class of  $X \in \mathcal{M}$ .

(a) Show that

$$\begin{aligned} d: (\mathcal{M}/\sim) \times (\mathcal{M}/\sim) &\rightarrow \mathbb{R} \\ ([X], [Y]) &\mapsto E[|X - Y| \wedge 1] \end{aligned}$$

is a metric on  $\mathcal{M}/\sim$ .

(b) Let  $(X_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{M}$  and let  $X$  be an element of  $\mathcal{M}$ . Show that  $([X_n])_{n \in \mathbb{N}}$  converges to  $[X]$  with respect to the metric  $d$  if and only if  $(X_n)_{n \in \mathbb{N}}$  converges to  $X$  in probability.

**Exercise 3.3** Let  $X_i, i \geq 1$ , be identically distributed, integrable random variables and define  $S_n = \sum_{i=1}^n X_i$  for each  $n \in \mathbb{N}$ . Show that:

$$\lim_{M \rightarrow \infty} \sup_{n \geq 1} E \left[ \frac{|S_n|}{n} 1_{\left\{ \frac{|S_n|}{n} > M \right\}} \right] = 0.$$

*Note:* This family  $\left\{ \frac{|S_n|}{n}, n \in \mathbb{N} \right\}$  is thus so-called “uniformly integrable”. See (3.6.14) in the lecture notes. Thanks to Theorem 3.41 and the strong law of large numbers, one has that: if  $X_i, i \geq 1$ , are also pairwise independent, (in addition to being identically distributed as in the question), then  $\frac{S_n}{n}$  converges  $P$ -a.s. and in  $L^1$  towards  $E[X_1]$  for  $n \rightarrow \infty$ .

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**Submission deadline:** 13:15, Oct 16

**Location:** During exercise class or in the tray outside of HG G53–54.

**Class assignment:**

Students	Time & Date	Room	Assistant
An-Gu	Tue 13-14	HG F 26.5	Daniel Balint
Ha-Lang	Tue 13-14	ML H 41.1	Daniel Contreras Salinas
Lanz-Sa	Tue 14-15	HG F 26.5	Daniel Balint
Sch-Zh	Tue 14-15	ML H 41.1	Chong Liu

**Office hours (Präsenz);** Mon. and Thu., 12:00 - 13:00 in HG G32.6.

Exercise sheets and further information are also available on:

<http://metaphor.ethz.ch/x/2018/hs/401-3601-00L/>