Probability Theory

Exercise sheet 4

Exercise 4.1

(a) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real random variables converging in probability to a random variable X. Show that $(X_n)_{n\in\mathbb{N}}$ converges to X in distribution.

(b) The converse does not hold in general. Instead, show that if the sequence $(X_n)_{n\in\mathbb{N}}$ converges in distribution to a *constant* random variable $X = c$, then $(X_n)_{n\in\mathbb{N}}$ converges in probability to c.

Exercise 4.2 (A version of the Glivenko-Cantelli Theorem)
Let $(X_i)_{i\in\mathbb{N}}$ be real-valued, i.i.d. random variables on $(\Omega, \mathcal{A}, P)$ with continous distribution function $F : \bar{\mathbb{R}} \to [0, 1]$. Show that for the empirical distribution

$$\begin{aligned} F_n : \bar{\mathbb{R}} &\to [0,1] \\ x &\mapsto \frac{1}{n}\sum_{i=1}^{n} 1_{\{X_i \leq x\}} \end{aligned}$$

holds that,

$$\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \overset{n\to\infty}{\longrightarrow} 0 \quad P\text{ -a.s.}$$

Hint: Show as an intermediate step that for every continous and non-decreasing function $F : \bar{\mathbb{R}} \to [0, 1]$ and every sequence $(F_n : \bar{\mathbb{R}} \to [0, 1])_{n\in\mathbb{N}}$ of non-decreasing functions holds that if $F_n(x) \overset{n\to\infty}{\longrightarrow} F(x)$ for all $x \in \bar{\mathbb{Q}} = \mathbb{Q} \cup \{\pm\infty\}$, then $(F_n)_{n\in\mathbb{N}}$ converge uniformly to F.
Remark: The statement of Glivenko-Cantelli also holds for non-continous distribution functions as well.

Exercise 4.3

(a) Let f be a (not necessarily Borel-measurable) function from $\mathbb{R}$ to $\mathbb{R}$. Show that the set of discontinuities of f, defined as

$$U_f := \left\{x \in \mathbb{R} \mid f \text{ is discontinuous in } x\right\},$$

is Borel-measurable.

(b) Assume that $X_n \to X$ in distribution. Let f be measurable and bounded, such that $P[X \in U_f] = 0$. Use (2.2.13) – (2.2.14) from the lecture notes to show that we have

$$E[f(X_n)] \underset{n\to\infty}{\to} E[f(X)].$$

(c) Let f be measurable and bounded on $[0, 1]$, with U_f of Lebesgue measure 0. Show that the corresponding Riemann sums converge to the integral of f, i.e.

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \underset{n\to\infty}{\to} \int_0^1 f(x)\mathrm{d}x.$$

Submission deadline: 13:15, Oct 23
Location: During exercise class or in the tray outside of HG G53–54.
Class assignment:

Students	Time & Date	Room	Assistant
An-Gu	Tue 13-14	HG F 26.5	Daniel Balint
Ha-Lang	Tue 13-14	ML H 41.1	Daniel Contreras Salinas
Lanz-Sa	Tue 14-15	HG F 26.5	Daniel Balint
Sch-Zh	Tue 14-15	ML H 41.1	Chong Liu

Office hours (Präsenz); Mon. and Thu., 12:00 - 13:00 in HG G32.6.
Exercise sheets and further information are also available on:
http://metaphor.ethz.ch/x/2018/hs/401-3601-00L/