Probability Theory

Exercise sheet 5

Exercise 5.1 Compute the characteristic functions of the following distributions:

(a) The triangular distribution $(1 - |x|)1_{[-1,1]}(x)\mathrm{d}x$.

(b) The Cauchy distribution $\frac{\alpha}{\pi} \frac{1}{x^2+\alpha^2}\mathrm{d}x$ with parameter $\alpha > 0$.
 Hint: Use a contour integral.

Exercise 5.2 Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables in a probability space $(\Omega, \mathcal{A}, P)$. Define the two sequences of random variables $(Y_n)_{n\in\mathbb{N}}$ and $(M_n)_{n\in\mathbb{N}}$ by

$$Y_n := \min_{1\leq i\leq n} X_i \quad \text{and} \quad M_n := \max_{1\leq i\leq n} X_i$$

(a) Let X_1 be uniformly distributed on the interval $[0,1]$. Show that nY_n converges in distribution to an exponential random variable Z with parameter 1, i.e., the density of Z is $e^{-x}1_{[0,\infty)}(x), x \in \mathbb{R}$.

(b) Let X_1 be exponentially distributed with parameter 1. Show that $M_n - \log n$ converges in distribution to a random variable Z with Gumbel distribution, i.e. the density of Z is $e^{-x}\exp(-e^{-x}), x \in \mathbb{R}$.

Exercise 5.3 Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of probability measures with

$$\int_{\mathbb{R}} x^k\, P_n(dx) \stackrel{n\to\infty}{\longrightarrow} \alpha_k \in \mathbb{R} \qquad \text{for all } k \in \mathbb{N}.$$

Assume that there exists *exactly one* probability measure P with the k-th moment $\alpha_k = \int_{\mathbb{R}} x^k\, P(dx)$. Show that $(P_n)_{n\in\mathbb{N}}$ converges weakly towards P.
 Hint: First show that $(P_n)_{n\in\mathbb{N}}$ is tight. Then show that each subsequence of $(P_n)_{n\in\mathbb{N}}$ has a sub-subsequence that converges weakly towards P.
 Remark: Note that in general, there is no uniqueness of such P. A counter-example would be $Y = e^X$ with X a standard normal random variable.

Submission deadline: 13:15, Oct 30
Location: During exercise class or in the tray outside of HG G53–54.
Class assignment:

Students	Time & Date	Room	Assistant
An-Gu	Tue 13-14	HG F 26.5	Daniel Balint
Ha-Lang	Tue 13-14	ML H 41.1	Daniel Contreras Salinas
Lanz-Sa	Tue 14-15	HG F 26.5	Daniel Balint
Sch-Zh	Tue 14-15	ML H 41.1	Chong Liu

Office hours (Präsenz); Mon. and Thu., 12:00 - 13:00 in HG G32.6.
Exercise sheets and further information are also available on:
http://metaphor.ethz.ch/x/2018/hs/401-3601-00L/