Coordinator
Chong Liu

# Probability Theory

## Exercise sheet 6

**Exercise 6.1** Let $(X_i)_{i\geq 1}$ be i.i.d. with symmetric stable distribution of parameter $\alpha \in (0,2)$, see lecture notes p. 63.

(a) Find the distribution of $n^{-1/\alpha}(X_1 + \cdots + X_n)$.

(b) Does $\frac{1}{\sqrt{n}}(X_1 + \cdots + X_n)$ converge in distribution?

**Exercise 6.2** Prove that: If $n \geq 1$, $\{X_j\}_{j=1,\dots,n}$ are random variables and we denote by $\phi_j$ the characteristic function of $X_j$, then $\{X_j\}_{j=1,\dots,n}$ are independent if and only if

$$E\left[\exp\left\{i\sum_{j=1}^{n}\xi_j X_j\right\}\right] = \prod_{j=1}^{n}\phi_j(\xi_j),$$

for all $\xi_1, \dots, \xi_n \in \mathbb{R}$.
**Hint:** For $d \geq 1$, and $\nu$ a probability measure on $\mathbb{R}^d$, one can define the characteristic function $\phi_\nu : \mathbb{R}^d \to \mathbb{R}$ of $\nu$, as

$$\phi_\nu(\lambda) = \int_{\mathbb{R}^d} \exp(i\lambda \cdot x)\nu(dx),$$

where $\lambda \cdot x$ denotes the scalar product in $\mathbb{R}^d$, and then use (without proof) the following uniqueness property of characteristic functions of $\mathbb{R}^d$-valued random variables: if $\nu$ and $\mu$ are probability measures on $\mathbb{R}^d$ with the same characteristic function, then $\nu = \mu$, (cf. (2.3.13) the uniqueness property for one-dimensional random variables in the lecture notes).

**Exercise 6.3**

(a) Let $Z_n, n \geq 1$, and $Y_n, n \geq 1$, be random variables defined on the same probability space, and assume $Z_n \xrightarrow{d} Z$ (i.e. convergence in distribution) and $Y_n \xrightarrow{P} 0$. Show that $Z_n + Y_n \xrightarrow{d} Z$

**Hint:** Recall the proof of Exercise 4.1 (a).

(b) (*Random Index Central Limit Theorem*) Let $X_i$, $i \in \mathbb{N}$, be i.i.d. random variables with $E[X_i] = 0$, $E[X_i^2] = \sigma^2 \in (0,\infty)$. Furthermore, let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that $a_n \in \mathbb{N}$ for all $n$ and $a_n \to \infty$, and $(N_n)_{n\in\mathbb{N}}$ a sequence of $\mathbb{N}$–valued random variables, such that $N_n/a_n \to 1$ in probability. Let $S_n := \sum_{i=1}^{n} X_i$ for $n \in \mathbb{N}$. Show that

$$\frac{S_{N_n}}{\sigma\sqrt{a_n}} \text{ converges to } \mathcal{N}(0,1) \text{ in distribution.}$$

**Hint:** Show that $\frac{S_{a_n}}{\sigma\sqrt{a_n}}$ converges in distribution to $\mathcal{N}(0,1)$ and, using Kolmogorov's inequality, that

$$\frac{S_{N_n}}{\sigma\sqrt{a_n}} - \frac{S_{a_n}}{\sigma\sqrt{a_n}} \xrightarrow[n\to\infty]{P} 0.$$

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**Submission deadline**: 13:15, Nov 06
**Location**: During exercise class or in the tray outside of HG G53–54.

**Class assignment**:

| Students | Time & Date | Room | Assistant |
|---|---|---|---|
| An-Gu | Tue 13-14 | HG F 26.5 | Daniel Balint |
| Ha-Lang | Tue 13-14 | ML H 41.1 | Daniel Contreras Salinas |
| Lanz-Sa | Tue 14-15 | HG F 26.5 | Daniel Balint |
| Sch-Zh | Tue 14-15 | ML H 41.1 | Chong Liu |

**Office hours (Präsenz)**; Mon. and Thu., 12:00 - 13:00 in HG G32.6.
Exercise sheets and further information are also available on:
**http://metaphor.ethz.ch/x/2018/hs/401-3601-00L/**