ETH Zürich, HS 2018
D-MATH, D-PHYS
Prof. Dr. A-S Sznitman

Coordinator
Chong Liu

Probability Theory

Exercise sheet 8

Exercise 8.1 Let $n \geq 2$, and let $X_1, \ldots, X_n$ be i.i.d. random variables defined on a probability space $(\Omega, \mathcal{A}, P)$.

(a) Show that for every Borel function $g : \mathbb{R}^n \to \mathbb{R}$ with $E[|g(X_1, \ldots, X_n)|] < \infty$ and any permutation π of $\{1, \ldots, n\}$,

$$E[g(X_1, \ldots, X_n)] = E[g(X_{\pi(1)}, \ldots, X_{\pi(n)})].$$

(b) Set $S := X_1 + \ldots + X_n$ and assume that X_1 is integrable. Find a representation of $E[X_1|S]$ as a function of S.
Hint: First show that $E[X_1|S] = E[X_2|S]$ P-a.s.

Exercise 8.2 (Polya's Urn)
An urn initially contains s black and w white balls. We consider the following process. At each step a random ball is drawn from the urn, and is replaced by t balls of the same colour, for some fixed $t \geq 1$. We define the random variable Y_n as the proportion of black balls in the urn after the n-th iteration. Show that $E[Y_{n+1}|\sigma(Y_1, Y_2, \ldots Y_n)] = Y_n$, for all $n \in \mathbb{N}$, that is, $\{Y_n\}_{n\in\mathbb{N}}$ is a martingale.

Exercise 8.3 Let $A = (a_{kl})_{1\leq k,l\leq n} \in \mathbb{R}^{n\times n}$ be a symmetric positive definite $n \times n$ matrix, for $n > 1$. The n–dimensional normal distribution on $\mathcal{B}(\mathbb{R}^n)$ with covariance matrix A^{-1} is defined by

$$\nu_A(\mathrm{d}x) = (2\pi)^{-\frac{n}{2}} \sqrt{\det A}\, \exp\left(-\frac{1}{2} x^T A x\right) \lambda^n(\mathrm{d}x),$$

where λ^n is the Lebesgue measure on $\mathbb{R}^n$. Let $(X_1, \ldots, X_n)$ be random variables with joint distribution ν_A, and

$$\mathcal{F} := \sigma(X_k \mid 2 \leq k \leq n).$$

Show that $E[X_1|\mathcal{F}]$ equals a linear combination $\sum_{k=2}^n \alpha_k X_k$ (P-almost surely). Compute the coefficients α_k.

Hint: Consider the decompositions

$$A = \left(\begin{array}{c|c} a_{11} & b^T \\ \hline b & B \end{array}\right).$$

and

$$X = (X_1, \ldots, X_n) = (X_1, Y)$$

with $Y = (X_2, \ldots, X_n)$. Then for any $D := Y^{-1}(G)$ with $G \in \mathcal{B}(\mathbb{R}^{n-1})$ try to use the above decomposition of matrix A to show that $E[X_1 \mathbf{1}_D] = E[f(Y)\mathbf{1}_D]$, where $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is a linear mapping.

Submission deadline: 13:15, Nov 20
Location: During exercise class or in the tray outside of HG G53–54.

Class assignment:

Students	Time & Date	Room	Assistant
An-Gu	Tue 13-14	HG F 26.5	Daniel Balint
Ha-Lang	Tue 13-14	ML H 41.1	Daniel Contreras Salinas
Lanz-Sa	Tue 14-15	HG F 26.5	Daniel Balint
Sch-Zh	Tue 14-15	ML H 41.1	Chong Liu

Office hours (Präsenz); Mon. and Thu., 12:00 - 13:00 in HG G32.6.
Exercise sheets and further information are also available on:
http://metaphor.ethz.ch/x/2018/hs/401-3601-00L/