ETH Zürich, HS 2018
D-MATH, D-PHYS
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Coordinator
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Probability Theory

Exercise sheet 10

Exercise 10.1 Consider a Galton-Watson process (see p. 97 of the lecture notes) Z_n, $n \geq 0$, with offspring distribution $\nu = \text{Bin}(2, p)$, $p \in [0, 1]$. We are interested in the probability $\vartheta(p) = P[Z_n > 0, \ \forall n \geq 0]$ that the population does not go extinct. Show that

$$\vartheta(p) = \begin{cases} 0 & \text{if } 0 \leq p \leq 1/2; \\ \frac{2p-1}{p^2} & \text{if } 1/2 < p \leq 1. \end{cases}$$

Hint: One way to prove this is to use the results for the various cases (subcritical, critical, supercritical) from Section 3.5 A), pp. 97-101 of the lecture notes.

Exercise 10.2 An urn initially contains one red and one green ball. At each time $n = 1, 2, 3, \dots$ a random ball is drawn, and then is put back together with an additional ball of the same colour. Let Z_n be the number of red balls in the urn before the n-th ball is drawn (so that $Z_1 = 1$).

(a) Show that $\left(\frac{Z_n}{n+1}\right)_{n \in \mathbb{N}}$ is a martingale for the natural filtration.
Hint: Try to use Exercise 8.2.

(b) Show that $\left(\frac{Z_n}{n+1}\right)_{n \in \mathbb{N}}$ converges P-almost surely and in L^1 to a random variable X. Determine the distribution of X.
Hint: Recall that $\frac{1}{n} \sum_{k=1}^{n} \delta_{\frac{k}{n+1}} \xrightarrow{\text{d}} \lambda$, as $n \to \infty$, where λ is the Lebesgue measure on $[0, 1]$ and δ_x is the Dirac measure of x.

Exercise 10.3 Consider a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\{\mathcal{F}_n\}_{n \geq 0}$, and let X_n be an $\mathcal{F}_n$-martingale for which $|X_{n+1} - X_n| \leq M$ P-a.s. for some fixed $M < \infty$. Define the events C, D by

$$C := \{\lim X_n \text{ exists and is finite}\},$$
$$D := \{\limsup X_n = +\infty \text{ and } \liminf X_n = -\infty\}.$$

Show that $P[C \cup D] = 1$.
Hint: Show that $P[C^c \cap (\{\sup_{n \in \mathbb{N}} X_n < a\} \cup \{\inf_{n \in \mathbb{N}} X_n > -a\})] = 0$, for all $a > 0$, by considering the processes $\{X_{T_A \wedge n}\}_{n \geq 0}$, for $A = [a, \infty)$ and $A = (-\infty, -a]$, where $T_A = \inf\{n \geq 0 : X_n \in A\}$.

Submission deadline: 13:15, Dec 04
Location: During exercise class or in the tray outside of HG G53–54.
Class assignment:

Students	Time & Date	Room	Assistant
An-Gu	Tue 13-14	HG F 26.5	Daniel Balint
Ha-Lang	Tue 13-14	ML H 41.1	Daniel Contreras Salinas
Lanz-Sa	Tue 14-15	HG F 26.5	Daniel Balint
Sch-Zh	Tue 14-15	ML H 41.1	Chong Liu

Office hours (Präsenz); Mon. and Thu., 12:00 - 13:00 in HG G32.6.
Exercise sheets and further information are also available on:
http://metaphor.ethz.ch/x/2018/hs/401-3601-00L/