ETH Zürich, HS 2018
D-MATH, D-PHYS
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Coordinator
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# Probability Theory

## Exercise sheet 11

**Exercise 11.1** (The generalized Borel-Cantelli lemma)
Consider $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_n\}_{n\geq 0}$, and let $A_n \in \mathcal{F}_n$, $n \geq 1$, be a sequence of events. Show that, up to a $P$-nullset,

$$\limsup_{n\to\infty} A_n = \{\sum_{n\geq 1} P[A_n|\mathcal{F}_{n-1}] = \infty\}.$$

**Hint:** Use Exercise 10.3.

**Exercise 11.2** (Probabilistic solution to the discrete Dirichlet problem)
Let $A \subseteq \mathbb{Z}^d$ be finite, $f : \mathbb{Z}^d \setminus A \to \mathbb{R}$ any function, and $(S_n)_{n\in\mathbb{N}}$ a simple random walk on $\mathbb{Z}^d$ with starting point $S_0 = 0$. For $x \in \mathbb{Z}^d$ let $T_x := \inf\{n \geq 0; | \ x + S_n \notin A\}$. Finally, let $\mathcal{F}_n := \sigma(S_0, \ldots, S_n)$ and $g(x) := E[f(x + S_{T_x})]$.

(a) Show that $T_x < \infty$ $P$-a.s. Thus $f(x + S_{T_x})$ exists a.s.
**Hint:** Use Exercise 10.3.

(b) Show that $g$ solves the discrete Dirichlet problem on $A$ with boundary condition $f$, i.e.,

$$g(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{Z}^d \setminus A \\ \dfrac{1}{2d} \displaystyle\sum_{\substack{\|y-x\|=1 \\ y\in\mathbb{Z}^d}} g(y) & \text{if } x \in A \ . \end{cases}$$

(c) Show that $E\left[f(x + S_{T_x})\middle|\mathcal{F}_1\right] = g(x + S_{T_x\wedge 1})$ $P$-a.s.

**Exercise 11.3** Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of independent, non-negative random variables with expectation 1. Consider the natural filtration $(\mathcal{F}_n)_{n\geq 0}$. We define

$$M_0 = 1, \quad M_n = Y_1 Y_2 \cdots Y_n, \text{ for } n \in \mathbb{N}.$$

(a) Prove that $(M_n)_{n\in\mathbb{N}}$ is a non-negative martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$ and there exists a random variable $M_\infty$, so that $M_n \to M_\infty$ a.s.

(b) Let $a_n := E[\sqrt{Y_n}]$. Show that $a_n \in (0, 1]$.

(c) Show that if $\prod_n a_n > 0$, it holds that $M_n \to M_\infty$ in $L^1$ and $E[M_\infty] = 1$.
**Hint:** Let $\hat{Y}_n := \sqrt{Y_n}/a_n$ and $\hat{M}_n := \hat{Y}_1 \hat{Y}_2, \ldots, \hat{Y}_n$ for $n \geq 1$, $\hat{M}_0 = 1$. Note that $M_n \leq \hat{M}_n^2$ for $n \in \mathbb{N}$. Then use (a) together with Doob's inequality to conclude the proof.

(d) Show that if $\prod_n a_n = 0$, then $M_\infty = 0$ a.s.

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**Submission deadline**: 13:15, Dec 11
**Location**: During exercise class or in the tray outside of HG G53–54.
**Class assignment**:

| Students | Time & Date | Room | Assistant |
|---|---|---|---|
| An-Gu | Tue 13-14 | HG F 26.5 | Daniel Balint |
| Ha-Lang | Tue 13-14 | ML H 41.1 | Daniel Contreras Salinas |
| Lanz-Sa | Tue 14-15 | HG F 26.5 | Daniel Balint |
| Sch-Zh | Tue 14-15 | ML H 41.1 | Chong Liu |

**Office hours (Präsenz)**; Mon. and Thu., 12:00 - 13:00 in HG G32.6.
Exercise sheets and further information are also available on:
**http://metaphor.ethz.ch/x/2018/hs/401-3601-00L/**