

Probability Theory

Exercise sheet 12

Exercise 12.1 Let $(X_n)_{n \geq 0}$ be a sequence of random variables with values in $[0, 1]$. We set $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$. Suppose that $X_0 = a \in [0, 1]$ and

$$P \left[X_{n+1} = \frac{X_n}{2} \mid \mathcal{F}_n \right] = 1 - X_n, \quad P \left[X_{n+1} = \frac{1 + X_n}{2} \mid \mathcal{F}_n \right] = X_n.$$

- (a) Show that $(X_n)_{n \geq 0}$ is a \mathcal{F}_n -martingale that converge to a random variable X_∞ P -almost surely and in L^2 .
- (b) Show that $E[(X_{n+1} - X_n)^2] = \frac{1}{4}E[X_n(1 - X_n)]$.

Exercise 12.2 Let $(X_n)_{n \geq 0}$ be a uniformly integrable family of random variables on (Ω, \mathcal{A}, P) .

- (a) Assume that X_n converges to a random variable X in distribution. Show that

$$E[X_n] \xrightarrow{n \rightarrow \infty} E[X].$$

Remark: Compare to (3.6.18)–(3.6.20), p. 112 of the lecture notes.

- (b) Assume that X_n converges to a random variable X in probability. Show that $X \in L^1$ and that X_n converges to X in L^1 .

Exercise 12.3 Let $(\Omega, \mathcal{F}, (P_x)_{x \in E})$ be a canonical (time-homogenous) Markov chain with a countable state space E , a transition kernel K , and canonical coordinates $(X_n)_{n \geq 0}$. The matrix

$$Q = (Q(x, y))_{x, y \in E} := (K(x, \{y\}))_{x, y \in E} = (P_x[X_1 = y])_{x, y \in E}$$

is then called the *transition matrix* of the Markov chain. For the meanings of notation P_x and transition kernel we refer to p. 146 – 147 in lecture notes.

Let E be a countable set, (S, \mathcal{S}) a measurable space, $(Y_n)_{n \geq 1}$ a sequence of i.i.d. S -valued random variables. We define a sequence $(X_n)_{n \geq 0}$ through $X_0 = x \in E$ and $X_{n+1} = \Phi(X_n, Y_{n+1})$, where $\Phi : E \times S \rightarrow E$ is a measurable map. Show that $(X_n)_{n \geq 0}$ induces a time-homogenous Markov chain and calculate the corresponding transition matrix.

Submission deadline: 13:15, Dec 18

Location: During exercise class or in the tray outside of HG G53–54.

Class assignment:

Students	Time & Date	Room	Assistant
An-Gu	Tue 13-14	HG F 26.5	Daniel Balint
Ha-Lang	Tue 13-14	ML H 41.1	Daniel Contreras Salinas
Lanz-Sa	Tue 14-15	HG F 26.5	Daniel Balint
Sch-Zh	Tue 14-15	ML H 41.1	Chong Liu

Office hours (Präsenz); Mon. and Thu., 12:00 - 13:00 in HG G32.6.

Exercise sheets and further information are also available on:

<http://metaphor.ethz.ch/x/2018/hs/401-3601-00L/>