ETH Zürich, HS 2018
D-MATH, D-PHYS
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Coordinator
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Probability Theory

Exercise sheet 13

Exercise 13.1 Let $(\Omega, \mathcal{F}, (P_x)_{x\in E})$ be a canonical time–homogeneous Markov chain with a countable state space E, canonical coordinate process $(X_n)_{n\geq 0}$ and transition matrix Q. Let $A \subset E$ and τ_A the first entrance time of A, i.e., $\tau_A := \inf\{n \geq 0 \,|\, X_n \in A\}$. Suppose that there exists $n \geq 1$ and $\alpha > 0$ such that for all $x \in A^c$,

$$Q^n(x, A) = \sum_{a\in A} Q^n(x, a) = \sum_{a\in A} P_x[X_n = a] \geq \alpha.$$

Show that for all $x \in E$ we have that $P_x(\tau_A < +\infty) = 1$.

Exercise 13.2 Let E be a countable set, and $(\Omega, \mathcal{F}, (P_x)_{x\in E})$ a canonical time–homogeneous Markov chain with state space E, canonical coordinate process $(X_n)_{n\geq 0}$ and transition matrix $Q = (Q(x, y))_{x,y\in E}$. Let $F \subset E$ and set $\tau_F := \inf\{n \geq 0 \,|\, X_n \in F\}$.

Let $f : E \to \mathbb{R}^+$ be a bounded function such that $f(x) \geq Qf(x)$ (resp. $=$) for all $x \in F^c$, where

$$Qf(x) := \int_\Omega f(X_1(\omega))P_x(\mathrm{d}\omega) = \sum_{y\in E} f(y)Q(x, y).$$

Show that $(f(X_{n\wedge\tau_F})_{n\geq 0})$ for all $x \in E$ is a positive P_x-supermartingale (resp. P_x-martingale) with respect to the canonical filtration $(\mathcal{F}_n)_{n\geq 0}$.

Exercise 13.3 Let $(\Omega, \mathcal{F}, (P_x)_{x\in\mathbb{Z}})$ be a canonical (time-homogenous) Markov chain with state space $\mathbb{Z}$, transition matrix Q, and canonical coordinate process $(X_n)_{n\geq 0}$. We assume that

$$\sum_{y\in\mathbb{Z}} y^2 Q(x, y) < +\infty \text{ for all } x \in \mathbb{Z},$$

and set $b(x) := E_x[X_1]$, $a(x) := \mathrm{Var}_x(X_1) = E_x[(X_1 - b(x))^2]$.

(a) Represent $b(x)$ and $a(x)$ with the help of the matrix Q.

(b) Show that

$$E_x[X_{n+1}] = E_x[b(X_n)], \quad \mathrm{Var}_x(X_{n+1}) = \mathrm{Var}_x(b(X_n)) + E_x[a(X_n)].$$

Submission deadline: 13:15, Dec **20**
Location: In the tray outside of HG G53–54.
Class assignment:

Students	Time & Date	Room	Assistant
An-Gu	Tue 13-14	HG F 26.5	Daniel Balint
Ha-Lang	Tue 13-14	ML H 41.1	Daniel Contreras Salinas
Lanz-Sa	Tue 14-15	HG F 26.5	Daniel Balint
Sch-Zh	Tue 14-15	ML H 41.1	Chong Liu

Office hours (Präsenz); Mon. and Thu., 12:00 - 13:00 in HG G32.6.
Exercise sheets and further information are also available on:
http://metaphor.ethz.ch/x/2018/hs/401-3601-00L/