ETH Zürich HS 2018
D-MATH, D-PHYS
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Probability Theory

Self evaluation quiz, November 08

Solution

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1. **Solution**: See Theorem 1.30 for the statement of Kolmogorov's 0-1 law. It can be applied to investigate convergence property of stochastic series of independent random variables, see Section 1.4.

2. **Solution**: See Theorem 2.18 for the statement of Continuity Theorem.
If μ_n is the centered normal distribution with variance n, then its characteristic function $\varphi_n(t)$ is equal to $\exp\{-\frac{n}{2}t^2\}$, see e.g. (2.3.12) in the lecture notes. Hence, we have
$$\lim_{n\to\infty} \varphi_n(t) = \begin{cases} 1, & \text{if } t = 0, \\ 0, & \text{otherwise.} \end{cases}$$
Since the limit function is not continuous, it is not the characteristic function of any distribution. Hence, by the contrapositive of (2.3.24) in Theorem 2.18 from the lecture notes, the sequence μ_n, $n \geq 1$ does not converge weakly.

3. **Solution**: See Theorem 1.37 for the statement of Three Series Theorem.
If $X_k = \frac{1}{k^\alpha} + \frac{Z_k}{k}$, for $k \geq 1$, where Z_k are i.i.d random variables with $P[Z_k = 1] = P[Z_k = -1] = \frac{1}{2}$ and $\alpha > 0$, then using the same notations as in Theorem 1.37 we choose $A = 2$ and define $Y_k := X_k 1_{\{|X_k| \leq A\}}$ for $k \geq 1$, we have $Y_k = X_k$ for all $k \geq 1$. It is straightforward to check that conditions i) and iii) in Theorem 1.37 are satisfied no matter which value α takes. Since $E[Y_k] = \frac{1}{k^\alpha}$, $\sum_{k\geq 1} E[Y_k]$ converges if and only if $\alpha > 1$. Hence, by Theorem 1.37 we can conclude that if $\alpha > 1$ then $\sum_{k\geq 1} X_k$ converges a.s., otherwise it diverges a.s. (cf. Exercise 3.1).

4. **Solution**: See Theorem 2.24 for the statement of Lindeberg-Feller Theorem. For the proof that Lindeberg-Feller Theorem implies Central Limit Theorem we refer you to Remark 2.25.