Coordinator
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Probability Theory

Solution sheet 1

Solution 1.1 Let $|x|$ be the Euclidean norm of $x = (x_1, x_2) \in \mathbb{R}^2$ such that $|x|^2 = x_1^2 + x_2^2$. Then the random variable Z is the mapping $Z(x) = |x|^2$. Let F_Z denote the distribution function of Z, i.e., for $y \in \mathbb{R}$, $F_Z(y) = \mathbb{P}[Z \leq y]$. Clearly, since $Z(x) = |x|^2 \geq 0$ for all $x \in \mathbb{R}^2$, we have $F_Z(y) = 0$ for all $y < 0$. Now let $y \geq 0$ and $B_R(0) := \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| \leq R\}$ be the (closed) ball centered at $0 \in \mathbb{R}^2$ with radius $R \geq 0$. By the definition of the probability measure $\mathbb{P}$ we have

$$\begin{aligned} F_Z(y) = \mathbb{P}[Z \leq y] &= \mathbb{P}[B_{\sqrt{y}}(0)] \\ &= \int_{B_{\sqrt{y}}(0)} \frac{1}{2\pi} \exp\{-\frac{1}{2}|x|^2\}\, dx. \end{aligned}$$

Invoking the polar coordinates $x = (x_1, x_2) = (r\cos\theta, r\sin\theta)$ with $r = |x| \geq 0$ and $\theta \in [0, 2\pi)$ as well as the relation $dx = r dr d\theta$, we can derive that

$$\begin{aligned} \int_{B_{\sqrt{y}}(0)} \frac{1}{2\pi} \exp\{-\frac{1}{2}|x|^2\}\, dx &= \int_0^{2\pi} \int_0^{\sqrt{y}} \frac{1}{2\pi} \exp\{-\frac{r^2}{2}\} r\, dr\, d\theta \\ &= \int_0^{\sqrt{y}} \exp\{-\frac{r^2}{2}\} r\, dr \\ &= -\exp\{-\frac{r^2}{2}\}\Big|_0^{\sqrt{y}} \\ &= 1 - \exp\{-\frac{y}{2}\}. \end{aligned}$$

Hence, the distribution function of Z is

$$F_Z(y) = \begin{cases} 1 - \exp\{-\frac{y}{2}\}, & \text{if } y \geq 0; \\ 0, & \text{if } y < 0. \end{cases}$$

In other words, Z has the **exponential distribution with parameter** $\frac{1}{2}$.

Solution 1.2

(a) By definition, $\sigma(\mathcal{Z})$ is the smallest σ-algebra that contains all A_i, $i \in I$, i.e.,

$$\sigma(\mathcal{Z}) := \bigcap_{\substack{\mathcal{U} : \mathcal{U} \text{ is a} \\ \sigma\text{-algebra} \\ \text{containing all } A_i}} \mathcal{U}. \tag{1}$$

We now show that $\sigma(\mathcal{Z}) = \left\{ \bigcup_{i \in J} A_i \,\middle|\, J \subseteq I \right\}$:

"$\supseteq$" For any σ-algebra $\mathcal{U}$ that contains all A_i it holds that:

$$\bigcup_{i \in J} A_i \in \mathcal{U}, \qquad J \subseteq I,$$

since J, being a subset of I, is countable, and σ-algebras are closed under countable, unions by definition. Therefore, we have that

$$\sigma(\mathcal{Z}) \overset{(1)}{=} \bigcap \mathcal{U} \supseteq \left\{ \bigcup_{i \in J} A_i \,\middle|\, J \subseteq I \right\}.$$

"$\subseteq$" Since $\mathcal{U}$ contains all A_i, it is sufficient to show that

$$\mathcal{U} = \left\{ \bigcup_{i \in J} A_i \,\middle|\, J \subseteq I \right\}$$

is a σ-algebra. We verify the conditions:

- $\bigcup_{i \in J} A_i = \Omega$, by choosing $J = I$, so $\Omega \in \mathcal{U}$,
- for any $J \subset I$, $\left(\bigcup_{i \in J} A_i\right)^c = \bigcup_{i \in I \setminus J} A_i \in \mathcal{U}$,
- if $J_n \subseteq I$, $n \geq 1$, then

$$\bigcup_{n \geq 1} \left(\bigcup_{i \in J_n} A_i \right) = \bigcup_{i \in \underbrace{\bigcup_{n \geq 1} J_n}_{=: J \subseteq I}} A_i \in \mathcal{U}.$$

(b) Let

$$F_1 := \left\{ f : \Omega \to \mathbb{R} \,\middle|\, f \text{ is } \sigma(\mathcal{Z})\text{-measurable} \right\} \text{ and}$$
$$F_2 := \left\{ f : \Omega \to \mathbb{R} \,|\, f \text{ is constant on } A_i, i \in I \right\}.$$

We want to show that $F_1 = F_2$:

"$\supseteq$" Let $f \in F_2$. Then we can write

$$f(x) = a_i \text{ for } x \in A_i,$$

for some $a_i \in \mathbb{R}$. To check that f is $\sigma(\mathcal{Z})$-measurable, it suffices to check that $\{x \in \Omega : f(x) \leq a\}$ is a measurable set for all $a \in \mathbb{R}$. So let $a \in \mathbb{R}$ and decompose I in two disjoint sets I_1, I_2 such that

- $a_i \leq a$ for all $i \in I_1$ and
- $a_i > a$ for all $i \in I_2$.

We then have

$$\{f \leq a\} = \bigcup_{i \in I_1} \{f = a_i\} = \bigcup_{i \in I_1} A_i \in \sigma(\mathcal{Z}).$$

"$\subseteq$" Let $f \in F_1$. If f is measurable then the pre-image under f of any Borel measurable subset of $\mathbb{R}$ must be measurable. Therefore $\{x \in \Omega : f(x) = a\} = f^{-1}(\{a\}) \in \sigma(\mathcal{Z})$ for all $a \in \mathbb{R}$. Thus, from part (a) we have $\{x \in \Omega : f(x) = a\} = \bigcup_{i \in J} A_i$ for some $J \subseteq I$. In particular, for all $i \in I$ and $a \in \mathbb{R}$

$$\{f = a\} \cap A_i \in \left\{\emptyset, A_i\right\},$$

which implies that f is constant on A_i and $f \in F_2$.

Solution 1.3

(a) If $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{R}$-$\mathcal{R}$-measurable, $Y = f \circ X$ and $B \in \mathcal{R}$ then

$$(f \circ X)^{-1}(B) = X^{-1}(\underbrace{f^{-1}(B)}_{\in \mathcal{R}}) \in \sigma(X).$$

That is Y is $\sigma(X)$-$\mathcal{R}$-measurable.

(b) Since $A \in \sigma(X)$, there is a $B \in \mathcal{R}$ such that $A = X^{-1}(B)$. Therefore

$$Y = 1_A = 1_{X^{-1}(B)} = 1_B \circ X,$$

so the $\implies$ direction holds for indicator functions.

(c) For each i we can apply part (b) to get a $B_i \in \mathcal{R}$ such that $1_{A_i} = 1_{B_i} \circ X$. Then

$$Y = \sum_{i=1}^{n}((c_i 1_{B_i}) \circ X) = (\sum_{i=1}^{n} c_i 1_{B_i}) \circ X = f \circ X,$$

with $f = \sum_{i=1}^{n}(c_i 1_{B_i})$. Furthermore f is $\mathcal{R}$-$\mathcal{R}$-measurable, so $\implies$ direction holds for linear combinations of indicator functions.

(d) Define the "step function approximations"

$$Y_n := \sum_{k=0}^{n2^n-1} \frac{k}{2^n} 1_{\left\{\frac{k}{2^n} \le Y < \frac{k+1}{2^n}\right\}} + n 1_{\{Y \ge n\}}.$$

We then have $Y_n \uparrow Y$. Also Y_n is a linear combination of indicator functions for all n, and since Y is $\sigma(X)$-$\mathcal{R}$-measurable the sets $\left\{\frac{k}{2^n} \le Y < \frac{k+1}{2^n}\right\} \subset \Omega$ are in $\sigma(X)$ (using also that $[k/2^n, (k+1)/2^n)$ and $[n, \infty)$ are in $\mathcal{R}$). Thus, from (c) we know that there are $\mathcal{R}$-$\mathcal{R}$-measurable functions f_n such that $Y_n = f_n \circ X$. We define

$$g(x) := \limsup_{n\to\infty} f_n(x).$$

Since the lim sup of a sequence of measurable functions is measurable, we have that g is a measurable function from $\mathbb{R}$ to $(-\infty, \infty]$. It can happen that $g(x) = \infty$ (but only for x outside the range of X), so to deal with this technicality we set

$$f(x) := 1_{\{g(x)<\infty\}} g(x), x \in \mathbb{R}.$$

Then f is $\mathcal{R}$-$\mathcal{R}$-measurable. Also, since $Y_n \uparrow Y$ we have that $f(x) = \lim_{n\to\infty} f_n(x)$ for x in the range of X, and thus

$$Y = \lim_{n\to\infty} Y_n = \lim_{n\to\infty} f_n \circ X = (\lim_{n\to\infty} f_n) \circ X = f \circ X.$$

This proves the $\implies$ direction for non-negative Y.

(e) Write

$$Y = Y^+ - Y^-,$$

for $Y^+ = 1_{Y\ge 0} Y$ and $Y^- = -1_{Y<0} Y$. Then d) applies to Y^+ and Y^-, so we have functions f and g such that

$$Y^+ = f \circ X \text{ and } Y^- = g \circ X.$$

Clearly

$$Y = (f - g) \circ X,$$

and $f - g$ is $\mathcal{R}$-$\mathcal{R}$-measurable, so the claim follows.