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Probability Theory

Solution sheet 2

Solution 2.1

(a) Let $\omega \in \Omega$. We will show

$$\omega \in \bar{A} \Leftrightarrow \limsup_{n\to\infty} 1_{A_n}(\omega) = 1. \tag{1}$$

The case on $\bar{A}^c$ is handled analogously.

Let $\limsup_{n\to\infty} 1_{A_n}(\omega) = 1$. Then for all $n \in \mathbb{N}$ we have

$$1 = \limsup_{m\to\infty} 1_{A_m}(\omega) = \inf_{m\in\mathbb{N}} \sup_{k\geq m} 1_{A_k}(\omega) \leq \sup_{k\geq n} 1_{A_k}(\omega) \leq 1,$$

and thus $\sup_{k\geq n} 1_{A_k}(\omega) = 1$. Since the indicator function 1 only takes the values 0 and 1, there exists for each $n \in \mathbb{N}$ a $k \geq n$ such that $\omega \in A_k$. In other words, $\omega \in \bar{A}$.

If on the other hand $\omega \in \bar{A}$, then there exists for all $n \in \mathbb{N}$ a $k \geq n$ for which $\omega \in A_k$. Thus $\sup_{k\geq n} 1_{A_k}(\omega) = 1$ for every $n \in \mathbb{N}$, implying $\limsup_{n\to\infty} 1_{A_n}(\omega) = 1$.

Thus, we have shown (1), and $1_{\bar{A}} = \limsup_{n\to\infty} 1_{A_n}$ follows. To show the analogous result for the lim inf we note that

$$(\underline{A})^c = \left(\bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k\right)^c = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k^c = \limsup_{n\to\infty} A_n^c.$$

Thus we can deduce the result for lim inf from the already proven result for lim sup.

$$\begin{aligned} 1_{\underline{A}} &= 1 - 1_{(\underline{A})^c} = 1 - 1_{\limsup_{n\to\infty} A_n^c} \\ &= 1 - \limsup_{n\to\infty} 1_{A_n^c} = \liminf_{n\to\infty} (1 - 1_{A_n^c}) = \liminf_{n\to\infty} 1_{A_n}. \end{aligned}$$

(b) These inequalities are immediate consequences of (a) and Fatou's lemma.

Solution 2.2

(a) We start with the first claim. Because $\{a\} = \{a,b\} \cap \{a,c\}$, we know that $\{a\} \in \sigma(\mathcal{C})$. By cyclic symmetry we obtain that $\{b\}, \{c\}, \{d\} \in \sigma(\mathcal{C})$ as well. The claim follows from Exercise 1.1 (a). For the second claim, we simply observe that $\forall B \in \mathcal{C}$, $P(B) = Q(B) = 1/2$.

(b) Suppose $\{A \in \mathcal{A};\ P(A) = Q(A)\}$ is a σ-algebra. Since this collection contains $\mathcal{C}$, by (a), it would contain also $\mathcal{A}$, by (a). Thus, P and Q would be equal, which is a contradiction.

(c) No. By a direct inspection, we see that

$$\{a,b\} \cap \{a,c\} = \{a\} \notin \mathcal{C}.$$

We can also show this by the following argument: If it were a π-system, by (1.3.11) in the lecture notes, any P and Q agreeing on $\mathcal{C}$ would be equal.

Solution 2.3 We claim that each $\omega \in \Omega$ can be written as

$$\omega = \sum_{j \geq 1} Y_j(\omega) 2^{-j}. \tag{2}$$

To see this, we write down the binary representation of ω, i.e.

$$\begin{aligned} \omega &= \sum_{j \geq 1} \omega_j 2^{-j}, \quad \omega_j \in \{0, 1\}, \\ &= 0.\omega_1 \omega_2 \ldots \end{aligned}$$

Technical point: In cases like $\omega = 1/2$, which can be represented as both $0.1000\ldots$ and $0.01111\ldots$, we choose the terminating binary representation, i.e the one which "ends" in an infinite sequence of zeroes, which is the usual convention.

$$\begin{aligned} \text{For } \omega = 0.\omega_1\omega_2 \ldots \omega_j \ldots \ &\Rightarrow 2^j \omega = \omega_1 \omega_2 \ldots \omega_j.\omega_{j+1} \ldots \in [\omega_1\omega_2 \ldots \omega_j,\ \omega_1\omega_2\ldots\omega_j + 1), \\ &\Rightarrow \lfloor 2^j \omega \rfloor = \omega_1 \ldots \omega_j \Rightarrow \begin{cases} \lfloor 2^j \omega \rfloor = \omega_1 \ldots \omega_j \text{ is odd}, & \Rightarrow \omega_j = 1, \\ \lfloor 2^j \omega \rfloor = \omega_1 \ldots \omega_j \text{ is even}, & \Rightarrow \omega_j = 0. \end{cases} \end{aligned}$$

Hence we have $Y_j(\omega) = \omega_j$. From the representation (2), we see that for $n \geq 1$

$$\{Y_n = 0\} = \Omega \cap \bigcup_{j=0}^{2^{n-1}-1} \left[\frac{2j}{2^n}, \frac{2j+1}{2^n} \right).$$

Thus the Y_n are measurable, and $P[Y_n = 0] = 2^{n-1}/(2^n) = 1/2 = P[Y_n = 1]$. To prove independence, we note that for $n \geq 1$ and $z_1, z_2, \ldots, z_n \in \{0, 1\}$, we have

$$P\left[\bigcap_{j=1}^{n} \{Y_j = z_j\} \right] = P\left[\left[\sum_{j=1}^{n} \frac{z_j}{2^j}, \sum_{j=1}^{n} \frac{z_j}{2^j} + \frac{1}{2^n} \right) \right] = 2^{-n} = \prod_{j=1}^{n} P[Y_j = z_j].$$

By the observation given in the hint, this implies independence of the infinite sequence $\{Y_n\}$, $n \geq 1$.