Coordinator
Chong Liu

Probability Theory

Solution sheet 3

Solution 3.1

Using the same notations as in the statement of Theorem 1.37 (Kolmogorov's three-series theorem), we choose $A = 2$ and define $Y_k := X_k 1_{\{|X_k| \leq A\}}$ for $k \geq 1$. From the definition of X_k and the fact that $|Z_k| \leq 1$ we actually have $|X_k| \leq 2$ and thus $Y_k = X_k$ for all $k \geq 1$.
Since $\mathrm{Var}(Z_k) = \frac{1}{2}$ for all $k \geq 1$, we have $\mathrm{Var}(X_k) = \mathrm{Var}(\frac{1}{k^2} + \frac{Z_k}{k^{\frac{1}{4}}}) = \frac{1}{k^{\frac{1}{2}}} \mathrm{Var}(Z_k) = \frac{1}{2k^{\frac{1}{2}}}$. Hence, it holds that $\sum_{k \geq 1} \mathrm{Var}(Y_k) = \sum_{k \geq 1} \frac{1}{2k^{\frac{1}{2}}} = \infty$, which implies that the condition iii) in (1.4.17) fails. By Theorem 1.37, we then obtain that $\sum_{k \geq 1} X_k$ cannot converge P-a.s., or in other words, $P[\sum_{k \geq 1} X_k \text{ converges}] < 1$. Since the event $\{\sum_{k \geq 1} X_k \text{ converges}\}$ belongs to the asymptotic σ-algebra $\mathcal{F}_\infty$ associated with independent random variables $X_k.k \geq 1$, by Theorem 1.30 (Kolmogorov's 0-1 law) we can conclude that $P[\sum_{k \geq 1} X_k \text{ converges}] = 0$.

Solution 3.2

(a) We verify the criteria for d to be a metric

1. It is clear that d is well-defined;
2. From the definition of d we know that $\forall X, Y \; d([X],[Y]) = d([Y],[X])$;
3. It also follows from the definition of d that $\forall X \; d([X],[X]) = 0$;
4. That $d([X],[Y]) = 0$ for $X, Y \in L^0$ implies $X = Y$ P-.a.s., which further implies $[X] = [Y]$ in $\mathcal{M}/\sim$;
5. To prove that $\forall X, Y, Z \in L^0 \; d([X],[Z]) \leq d([X],[Y]) + d([Y],[Z])$, it is sufficient to note that for all $a, b, c \in \mathbb{R}$,

$$|a - c| \wedge 1 \leq |a - b| \wedge 1 + |b - c| \wedge 1.$$

(b) Assume $d([X_n],[X]) \to 0$. With Chebyshev's inequality it follows that

$$P[|X_n - X| > \varepsilon] = P[|X_n - X| \wedge 1 > \varepsilon] \leq \frac{E[|X_n - X| \wedge 1]}{\varepsilon} \to 0.$$

For the converse, assume $P[|X_n - X| > \varepsilon] \to 0$ for each $\varepsilon > 0$. Then, it follows that

$$\begin{aligned} E[|X_n - X| \wedge 1] &\leq E[|X_n - X| \wedge 1, |X_n - X| < \varepsilon] \\ &\quad + E[|X_n - X| \wedge 1, |X_n - X| \geq \varepsilon] \\ &\leq \varepsilon + P[|X_n - X| \geq \varepsilon] < 2\varepsilon, \end{aligned}$$

for sufficiently large n.

Solution 3.3 Let $\widetilde{S}_n = \sum_{i=1}^n |X_i|$. Since $\widetilde{S}_n \geq |S_n|$, we have,

$$1_{\left\{\frac{|S_n|}{n} > M\right\}} \leq 1_{\left\{\frac{\widetilde{S}_n}{n} > M\right\}}$$

which implies that

$$E\left[\frac{|S_n|}{n} 1_{\left\{\frac{|S_n|}{n} > M\right\}}\right] \leq E\left[\frac{\widetilde{S}_n}{n} 1_{\left\{\frac{\widetilde{S}_n}{n} > M\right\}}\right].$$

Hence we can assume, without loss of generality, that $X_i \geq 0$ for all i. Then we have that for $A > 0$:

$$\begin{aligned}
E\left[\frac{S_n}{n} 1_{\left\{\frac{S_n}{n} > M\right\}}\right] &= E\left[\frac{1}{n}\Big(\sum_{i=1}^{n} X_i 1_{\{X_i > A\}}\Big) 1_{\left\{\frac{S_n}{n} > M\right\}}\right] + E\left[\frac{1}{n}\Big(\sum_{i=1}^{n} X_i 1_{\{X_i \leq A\}}\Big) 1_{\left\{\frac{S_n}{n} > M\right\}}\right] \\
&\leq E\left[\frac{1}{n}\sum_{i=1}^{n} X_i 1_{\{X_i > A\}}\right] + E\left[\frac{1}{n}\sum_{i=1}^{n} A\, 1_{\left\{\frac{S_n}{n} > M\right\}}\right] \\
&= E\left[X_1 1_{\{X_1 > A\}}\right] + A\ P\left[\frac{S_n}{n} > M\right] \\
&\overset{(*)}{\leq} E\left[X_1 1_{\{X_1 > A\}}\right] + \frac{A}{M}\, E\left[\frac{S_n}{n}\right] \\
&= E\left[X_1 1_{\{X_1 > A\}}\right] + \frac{A}{M}\, E[X_1],
\end{aligned}$$

where we have used the fact that $X_i \geq 0$ for all i and applied Chebyshev's inequality (1.2.13) at $(*)$.

Now we take $A = \sqrt{M}$. Then:

$$\overline{\lim_{M\to\infty}} \sup_{n\geq 1} E\left[\frac{S_n}{n} 1_{\{\frac{S_n}{n} > M\}}\right] \leq \overline{\lim_{M\to\infty}}\, E\left[X_1 1_{\{X_1 > \sqrt{M}\}}\right] + \overline{\lim_{M\to\infty}} \frac{1}{\sqrt{M}}\, E[X_1] = 0.$$

Where the last equality follows by dominated convergence and the fact that X_1 is integrable.