ETH Zürich, HS 2018
D-MATH, D-PHYS
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Probability Theory

Solution sheet 4

Solution 4.1

(a) We denote the distribution functions of X_n and X by F_n and F respectively. We have to show that $\lim_{n\to\infty} F_n(y) = F(y)$ for continuity points y of F.

So, let $y \in \mathbb{R}$ be a continuity point of F and let $\varepsilon > 0$. By the continuity of F in y there is a $\delta > 0$ such that

$$F(y) - \varepsilon \leq F(x) \leq F(y) + \varepsilon, \quad x \in [y-\delta, y+\delta]. \tag{1}$$

Since the X_n converge to X in probability, there is a $N \in \mathbb{N}$ such that

$$P\left[|X_n - X| > \delta\right] \leq \varepsilon, \quad n \geq N. \tag{2}$$

Now, for $n \geq N$,

$$\begin{aligned} F_n(y) = P[X_n \leq y] &\leq P[\{X \leq y + \delta\} \cup \{|X - X_n| > \delta\}] \\ &\leq P[X \leq y+\delta] + P[|X - X_n| > \delta] \\ &\overset{(2)}{\leq} F(y+\delta) + \varepsilon \overset{(1)}{\leq} F(y) + 2\varepsilon \end{aligned}$$

and

$$\begin{aligned} F_n(y) = P[X_n \leq y] &\geq P[\{X \leq y - \delta\} \setminus \{|X - X_n| > \delta\}] \\ &\geq P[X \leq y-\delta] - P[|X - X_n| > \delta] \\ &\overset{(2)}{\geq} F(y-\delta) - \varepsilon \overset{(1)}{\geq} F(y) - 2\varepsilon, \end{aligned}$$

so that

$$F(y) - 2\varepsilon \leq F_n(y) \leq F(y) + 2\varepsilon.$$

Thus,

$$F(y) - 2\varepsilon \leq \liminf_{n\to\infty} F_n(y) \leq \limsup_{n\to\infty} F_n(y) \leq F(y) + 2\varepsilon.$$

But this holds for all $\varepsilon > 0$, so we are done.

(b) We assume that for a $c \in \mathbb{R}$

$$X_n \to c \text{ in distribution.} \tag{3}$$

The constant c has distribution function

$$F(x) = 1_{[c,\infty)},$$

which is continuous except in c. So we know from (3)

$$F_n(z) = P[X_n \leq z] \to \begin{cases} 0 & \text{if } z < c, \\ 1 & \text{if } z > c. \end{cases} \tag{4}$$

We want to show that for all $\varepsilon > 0$

$$\lim_{n\to\infty} P[|X_n - c| \geq \varepsilon] = 0.$$

Now,

$$P[|X_n - c| \geq \varepsilon] = P[X_n \leq c - \varepsilon] + P[X_n \geq c + \varepsilon],$$

so that

$$\lim_{n\to\infty} P[|X_n - c| \geq \varepsilon] \leq \lim_{n\to\infty} P[X_n \leq c - \varepsilon] + \lim_{n\to\infty} P[X_n \geq c + \varepsilon].$$

By (4), we have $\lim_{n\to\infty} P[X_n \leq c - \varepsilon] = 0$ for all $\varepsilon > 0$. Furthermore

$$P[X_n \geq c + \varepsilon] = 1 - P[X_n < c + \varepsilon] \leq 1 - P\left[X_n \leq c + \frac{\varepsilon}{2}\right].$$

Thus,

$$\lim_{n\to\infty} P[X_n \geq c + \varepsilon] \leq 1 - \lim_{n\to\infty} P\left[X_n \leq c + \frac{\varepsilon}{2}\right] \overset{(4)}{=} 0.$$

Solution 4.2

We start proving the statement of the hint. Let $\epsilon > 0$ and take $-\infty = x_0 < x_1 < \cdots < x_n = \infty \in \bar{\mathbb{Q}} = \mathbb{Q} \cup \{\pm\infty\}$ for some $n \in \mathbb{N}$ such that

$$F(x_{i+1}) - F(x_i) \leq \epsilon, \tag{5}$$

note that this is possible by the continuity of F and the fact that F it is non-decreasing with $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = 0$. We assume that $F_m(x) \overset{m\to\infty}{\to} F(x)$ for all $x \in \bar{\mathbb{Q}}$, which imply that there exists $N \in \mathbb{N}$ such that for all $m \geq N$

$$\sup_{0\leq i\leq n} |F_m(x_i) - F(x_i)| < \epsilon. \tag{6}$$

Note that for all $x \in \bar{\mathbb{R}}$ there exists $i \in \{0, \ldots, n-1\}$ such that $x_i \leq x \leq x_{i+1}$. Combining (5) and (6) we get that,

$$F_m(x) - F(x) \leq F_m(x_{i+1}) - [F(x_{i+1}) - \epsilon] \leq 2\epsilon.$$

and

$$F(x) - F_m(x) \leq [F(x_i) + \epsilon] - F_m(x_i) \leq 2\epsilon.$$

Therefore $|F_m(x) - F(x)| \leq 2\epsilon$ for all $x \in \bar{\mathbb{R}}$ and $m \geq N$, so we get the desired uniform convergence. Now we apply this result to our problem. Let us remark that for each given $x \in \bar{\mathbb{R}}$, $F_n(x)$ is actually a random variable: $\omega \mapsto \frac{1}{n}\sum_{i=1}^n 1_{\{X_i(\omega)\leq x\}}$. To keep notation clean we usually omit ω in F_n.
By the Strong Law of Large Numbers,

$$F_n(x) = \frac{1}{n}\sum_{i=1}^n 1_{\{X_i\leq x\}} \to P(X_i \leq x) = F(x),\ P\text{-a.s.},$$

which imply that for all $x \in \bar{\mathbb{Q}}$ ($\pm\infty$ is trivial) there exists $N_x \subset \Omega$ with $P(N_x) = 0$ such that for all $\omega \notin N_x$, $F_n(x,\omega) \to F(x)$ as n goes to ∞. Let N be defined as $N := \bigcup_{y\in\bar{\mathbb{Q}}} N_y$, then for all $x \in \bar{\mathbb{Q}}$ and $\omega \in N$, $F_n(x,\omega) \to F(x)$ pointwise, therefore, by the hint above, $\forall\omega \notin N$, $F_n \to F$ uniformly as n goes to infinity. Since N is the countable union of set with measure 0 we have that $P(N) = 0$ and then $F_n(x) \to F(x)$ uniformly P-a.s..

Solution 4.3

(a) Let $f : \mathbb{R} \to \mathbb{R}$ and

$$V_{\epsilon,\delta} := \left\{x \in \mathbb{R} | \exists y, z \in (x-\delta, x+\delta) \text{ s.t. } |f(y) - f(z)| \geq \epsilon\right\}.$$

(i) Claim: $V_{\epsilon,\delta}$ is open.
Let $x \in V_{\epsilon,\delta}$. Then there are $y, z \in (x-\delta, x+\delta)$ such that $|f(y) - f(z)| \geq \epsilon$. We set $r := \delta - \max\{|y-x|, |z-x|\} > 0$.
$\Rightarrow \forall\, \tilde{x} \in (x-r, x+r)$ it holds that $|y - \tilde{x}| \leq |y-x| + |x-\tilde{x}| < |y-x| + r \leq \delta$, and similarly for z. From this it follows that $y, z \in (\tilde{x} - \delta, \tilde{x} + \delta)$ and $|f(y) - f(z)| \geq \epsilon$, which gives $\tilde{x} \in V_{\epsilon,\delta}$. So $(x-r, x+r) \subset V_{\epsilon,\delta}$, and the claim follows.

(ii) Claim: $U_f = \bigcup_n \bigcap_m V_{\frac{1}{n},\frac{1}{m}}$.
"$\subset$" Let $x \in U_f$. Then there is an $n \in \mathbb{N}$, such that

$$\forall\, m \in \mathbb{N}\ \exists\, y \in \left(x - \frac{1}{m}, x + \frac{1}{m}\right) \text{ s.t. } |f(x) - f(y)| \geq \frac{1}{n}.$$

"$\supset$" We assume that for some $n \in \mathbb{N}$, $x \in V_{\frac{1}{n},\frac{1}{m}}$, $\forall\, m$. Then there are $y, z \in (x - \frac{1}{m}, x + \frac{1}{m})$ so that $|f(y) - f(z)| \geq \frac{1}{n}$.
From this it follows that either $|f(y) - f(x)| \geq \frac{1}{2n}$ or $|f(z) - f(x)| \geq \frac{1}{2n}$ must hold. In other words $\exists\, n \in \mathbb{N}$, such that

$$\forall\, m \in \mathbb{N}\ \exists\, y \in \left(x - \frac{1}{m}, x + \frac{1}{m}\right) : |f(y) - f(x)| \geq \frac{1}{2n},$$

which implies that f is discontinuous in x.

Since the $V_{\frac{1}{n},\frac{1}{m}}$ are open, they are Borel measurable. And since any σ−algebra is closed under countable unions and intersections, $U_f = \bigcup_n \bigcap_m V_{\frac{1}{n},\frac{1}{m}}$ must also be Borel measurable.

(b) By (2.2.13) – (2.2.14) of the lecture notes, there exist $Y_n \overset{d}{=} X_n$, and $Y \overset{d}{=} X$, such that $Y_n \to Y$, P'-almost surely on a probability space $(\Omega', \mathcal{F}', P')$. Of course, we also have $f(Y_n) \overset{d}{=} f(X_n)$, and $f(Y) \overset{d}{=} f(X)$, so that we have $E[f(X_n)] = E'[f(Y_n)]$, and $E[f(X)] = E'[f(Y)]$, where we denote by E' the expectation w.r.t. P'. Thus, it suffices to show that

$$E'[f(Y_n)] \underset{n\to\infty}{\to} E'[f(Y)]. \tag{7}$$

Now, since $Y_n \to Y$, P'-almost surely, we have a set N, with $P'(N) = 0$, such that

$$\left\{\omega' \in \Omega' \mid Y_n(\omega') \underset{n\to\infty}{\to} Y(\omega')\right\} \cup N = \Omega'. \tag{8}$$

On the other hand, we have

$$\begin{aligned}\{\omega' \mid Y_n(\omega') \to Y(\omega')\} &\subseteq \{\omega' \mid f \text{ cont. in } Y(\omega'), Y_n(\omega') \to Y(\omega')\} \cup \{\omega \mid f \text{ discont. in } Y(\omega)\} \\ &\subseteq \{\omega' \mid f(Y_n(\omega')) \to f(Y(\omega'))\} \cup \{\omega' \mid Y(\omega') \in U_f\}.\end{aligned}$$

Consequently, it follows from equation (8) that we have

$$\{\omega' \mid f(Y_n(\omega')) \to f(Y(\omega'))\} \cup \{\omega' \mid Y(\omega') \in U_f\} \cup N = \Omega'.$$

But, by assumption $P'(Y \in U_f) = P(X \in U_f) = 0$ (recall that Y and X have the same distribution). Therefore, we get $f(Y_n) \to f(Y)$, P'-almost surely. Finally f is a bounded function, so by the Dominated Convergence Theorem equation (7) holds.

(c) Let λ be the Lebesgue measure on $[0,1]$, and, for all $a \in [0,1]$, let δ_a denote the Dirac delta measure on $[0,1]$. Let X_n, $n \geq 1$, be random variables with distribution $\frac{1}{n}\sum_{k=1}^{n} \delta_{\frac{k}{n}}$. Note that

$$E[f(X_n)] = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

Let X be a uniform random variable on $[0,1]$, hence it has distribution λ, and we note that

$$E[f(X)] = \int_0^1 f(x)\lambda(dx).$$

Thus, it suffices to show that we have

$$E[f(X_n)] \underset{n\to\infty}{\to} E[f(X)]. \tag{9}$$

Since by assumption $P[X \in U_f] = \lambda(U_f) = 0$, part **b)** implies that equation (9) is a consequence of the following:

$$X_n \stackrel{d}{\to} X. \tag{10}$$

To show equation (10), note that for all $n \in \mathbb{N}$,

$$P\left[X_n \leq a\right] = \begin{cases} 0, & a < 0, \\ \frac{[na]}{n}, & 0 \leq a \leq 1, \\ 1, & 1 < a. \end{cases}$$

Since we have $na - 1 < [na] \leq na$ (i.e. $[na]$ denotes the integer part of na), we get $\frac{[na]}{n} \underset{n\to\infty}{\to} a$, for all $0 \leq a \leq 1$. Thus, we obtain, for $0 \leq a \leq 1$,

$$P\left[X_n \leq a\right] \underset{n\to\infty}{\to} \lambda([0,a]) = P\left[X \leq a\right],$$

which implies equation (10), by definition. (Cases for $a < 0$ and $a > 1$ are trivially verified.)