Probability Theory

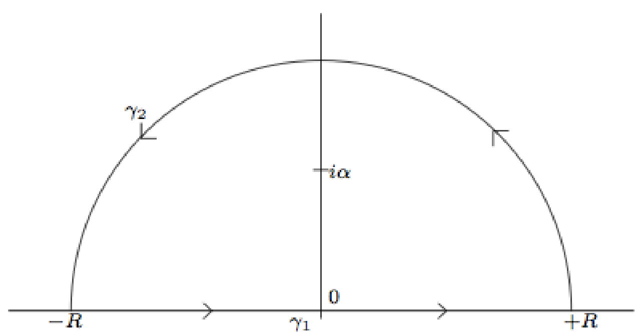
Solution sheet 5

Solution 5.1

(a) We calculate

$$\begin{aligned} E[e^{itX}] &= \int_{\mathbb{R}} e^{itx}(1-|x|)1_{[-1,1]}(x)\mathrm{d}x = \int_{-1}^{1}(1-|x|)e^{itx}\mathrm{d}x \\ &= \int_{-1}^{1} e^{itx}\mathrm{d}x + \int_{-1}^{0} xe^{itx}\mathrm{d}x - \int_{0}^{1} xe^{itx}\mathrm{d}x \\ &= \frac{1}{it}\left(e^{it}-e^{-it}\right) + \left(\frac{e^{-it}}{it} + \frac{1}{t^2} - \frac{e^{-it}}{t^2}\right) - \left(\frac{e^{it}}{it} + \frac{e^{it}}{t^2} - \frac{1}{t^2}\right) \\ &= \frac{2}{t}\sin(t) - \frac{2}{t}\sin(t) - \frac{2}{t^2}\cos(t) + \frac{2}{t^2} = \frac{2}{t^2}(1-\cos(t)). \end{aligned}$$

(b) We use the Residue Theorem (for more details on Residue theorem and contour integrals we refer you to the book "Real and complex analysis" by Rudin) for $f(x) = \frac{e^{itx}}{x^2+\alpha^2}$ to calculate $E\left[e^{itX}\right] = \frac{\alpha}{\pi}\int_{-\infty}^{+\infty} \frac{e^{itx}}{x^2+\alpha^2}\mathrm{d}x$. Extend f to $\mathbb{C}$ by $f(z) = \frac{e^{itz}}{z^2+\alpha^2}$. Let us consider the curve $\gamma = \gamma_1 \cup \gamma_2$ as follows:



Thus, we calculate for all $R > \alpha$,

$$\oint_{\gamma} f(z)\mathrm{d}z = 2\pi i \cdot \underbrace{n(\gamma, i\alpha)}_{=1} \cdot \mathrm{Res}_{i\alpha}(f) = 2\pi i \cdot \lim_{z\to i\alpha}(z-i\alpha)f(z) = 2\pi i \cdot \lim_{z\to i\alpha}\frac{e^{itz}}{z+i\alpha} = \frac{\pi}{\alpha}e^{-\alpha t}.$$

Since we have, for $t \geq 0$,

$$\begin{aligned}\left|\int_{\gamma_2} f(z)\mathrm{d}z\right| &= \left|\int_0^\pi f\left(Re^{i\theta}\right) iRe^{i\theta}\mathrm{d}\theta\right| = \left|\int_0^\pi \frac{e^{itRe^{i\theta}} iRe^{i\theta}}{R^2e^{2i\theta}+\alpha^2}\mathrm{d}\theta\right| \\ &\leq \int_0^\pi \left|\frac{e^{itRe^{i\theta}} iRe^{i\theta}}{R^2e^{2i\theta}+\alpha^2}\right|\mathrm{d}\theta = \int_0^\pi \frac{Re^{-Rt\sin(\theta)}}{|R^2e^{2i\theta}+\alpha^2|}\mathrm{d}\theta \\ &\leq \int_0^\pi \frac{R}{|R^2e^{2i\theta}+\alpha^2|}\mathrm{d}\theta \leq \int_0^\pi \frac{R}{R^2-\alpha^2}\mathrm{d}\theta = \frac{\pi R}{R^2-\alpha^2} \underset{R\to\infty}{\to} 0,\end{aligned}$$

we obtain that

$$E\left[e^{itX}\right] = \lim_{R\to\infty}\frac{\alpha}{\pi}\int_{\gamma_1} f(z)\mathrm{d}z = \lim_{R\to\infty}\frac{\alpha}{\pi}\int_{\gamma} f(z)\mathrm{d}z = e^{-\alpha t}.$$

For $t<0$, one can use a similar argument to show $E\left[e^{itX}\right] = e^{\alpha t}$. Thus, we get, for $t\in\mathbb{R}$,

$$E\left[e^{itX}\right] = e^{-\alpha|t|}.$$

Solution 5.2

(a) Take $y\in\mathbb{R}$. For $y<0$, because the exponential distribution is concentrated on $[0,\infty)$, we have that

$$P[nY_n\leq y] = 0 = P[Z\leq y] \text{ for all } n\in\mathbb{N}.$$

Hence without loss of generality we assume $y\geq 0$. It follows that for all $n\geq y$:

$$\begin{aligned}P\left[nY_n\leq y\right] &= P\left[Y_n\leq \frac{y}{n}\right] = P\left[\bigcup_{i=1}^n\left\{X_i\leq\frac{y}{n}\right\}\right] = 1-P\left[\bigcap_{i=1}^n\left\{X_i>\frac{y}{n}\right\}\right] \\ &= 1-\prod_{i=1}^n P\left[X_i>\frac{y}{n}\right] = 1-\left(1-\frac{y}{n}\right)^n.\end{aligned}$$

Now let $n\to\infty$, we hence obtain that

$$\lim_{n\to\infty} P[nY_n\leq y] = 1-e^{-y} = \int_{-\infty}^y e^{-x}1_{[0,\infty)}(x)\,\mathrm{d}x.$$

(b) Take $y\in\mathbb{R}$. Then for all $n\geq e^{-y}$ we have that:

$$\begin{aligned}P\left[M_n-\log n\leq y\right] &= P\left[M_n\leq y+\log n\right] = P\left[\bigcap_{i=1}^n\{X_i\leq y+\log n\}\right] \\ &= \prod_{i=1}^n \underbrace{P\left[X_i\leq y+\log n\right]}_{=1-e^{-y-\log n}=1-\frac{e^{-y}}{n}} = \left(1-\frac{e^{-y}}{n}\right)^n.\end{aligned}$$

Now let $n\to\infty$, we hence obtain that

$$\lim_{n\to\infty} P\left[M_n-\log n\leq y\right] = \exp(-e^{-y}) = \int_{-\infty}^y e^{-x}\exp(-e^{-x})\,\mathrm{d}x.$$

Solution 5.3 We first show that the claim in the hint implies the main claim. (cf. (2.3.25) in lecture notes). We assume that $P_n \stackrel{w}{\not\to} P$. Then there exists a point of continuity of $F(\cdot) := P((-\infty, \cdot])$ and a subsequence $n(k)$ with

$$|F_{n(k)}(y) - F(y)| \geq \epsilon \quad \forall\, k \in \mathbb{N}, \text{ where } F_n(\cdot) := P_n((-\infty, \cdot]).$$

Hence this subsequence has no sub-subsequence converging weakly to P. This contradicts obviously the claim in the hint.
To show the claim in the hint, we need multiple steps. We first show that $(P_n)_{n\in\mathbb{N}}$ is tight. Take $\epsilon > 0$ arbitrary. We obtain that

$$\begin{aligned} P_n\big([-M, M]^c\big) = \int_{-\infty}^{-M} P_n(dx) + \int_M^{\infty} P_n(dx) &\leq \int_{-\infty}^{-M} \frac{x^2}{M^2} P_n(dx) + \int_M^{\infty} \frac{x^2}{M^2} P_n(dx) \\ &\leq \frac{1}{M^2} \int_{-\infty}^{\infty} x^2 P_n(dx) \stackrel{n\to\infty}{\longrightarrow} \frac{\alpha_2}{M^2}. \end{aligned}$$

Because $\alpha_2 = \int_{\mathbb{R}} x^2 P(dx) < \infty$, we can choose an $M > 0$ such that $\frac{\alpha_2}{M^2} \leq \frac{\epsilon}{2}$. We can therefore find an $N \in \mathbb{N}$ with $P_n\big([-M, M]^c\big) \leq \epsilon \ \forall\, n > N$. On the other side it is clear that for each $k \in \{1, \ldots, N\}$, an $M_k > 0$ exists with $P_k\big([-M_k, M_k]^c\big) \leq \epsilon$. For $M^* := \max\{M, M_1, \ldots, M_N\}$ it follows then

$$\sup_{n\in\mathbb{N}} P_n\big([-M^*, M^*]^c\big) \leq \epsilon.$$

The tightness implies by (2.2.26) that each subsequence of $(P_n)_{n\in\mathbb{N}}$ has a sub-subsequence $(P_{n_{k(l)}})_{l\in\mathbb{N}}$ that converges weakly towards a probability measure $\widetilde{P}$.

We now show that $\widetilde{P} = P$. By Proposition 2.7 in the lecture notes, there exist random variables $Y_{n_{k(l)}}$, $l \in \mathbb{N}$ and $\widetilde{Y}$ in a probability space $(\Omega, \mathcal{A}, Q)$ such that

$$Y_{n_{k(l)}} \sim P_{n_{k(l)}},\ l \in \mathbb{N},\ \widetilde{Y} \sim \widetilde{P}, \text{ and } Y_{n_{k(l)}} \stackrel{l\to\infty}{\longrightarrow} \widetilde{Y}\ Q\text{-a.s.}$$

We show that the moments of $Y_{n_{k(l)}}$ for $l \to \infty$ converge towards those of $\widetilde{Y}$ respectively. Let $m \in \mathbb{N}$ and $M > 0$ arbitrary. Hence:

$$\begin{aligned} \Big| E^Q\big[Y^m_{n_{k(l)}}\big] - E^Q\big[\widetilde{Y}^m\big]\Big| &\leq E^Q\big[|Y^m_{n_{k(l)}} - \widetilde{Y}^m|\big] \\ &\leq E^Q\Big[|Y^m_{n_{k(l)}} - \widetilde{Y}^m|\, 1_{\{|Y^m_{n_{k(l)}}| \leq M,\, |\widetilde{Y}^m| \leq M\}}\Big] \\ &\quad + 3E^Q\Big[|Y^m_{n_{k(l)}}|\, 1_{\{|Y^m_{n_{k(l)}}| > M\}}\Big] + 3E^Q\Big[|\widetilde{Y}^m|\, 1_{\{|\widetilde{Y}^m| > M\}}\Big]. \end{aligned} \tag{1}$$

The first term converges towards 0 for $l \to \infty$ thanks to the Lebesgue dominated convergence theorem. For the second term, one has that

$$E^Q\Big[|Y^m_{n_{k(l)}}|\, 1_{\{Y^m_{n_{k(l)}} > M\}}\Big] = E^Q\left[\frac{Y^{2m}_{n_{k(l)}}}{|Y^m_{n_{k(l)}}|}\, 1_{\{|Y^m_{n_{k(l)}}| > M\}}\right] \leq \frac{1}{M} E^Q\big[Y^{2m}_{n_{k(l)}}\big] \stackrel{l\to\infty}{\longrightarrow} \frac{\alpha_{2m}}{M}.$$

Because we can take $M > 0$ arbitrarily big, we obtain that $E^Q\Big[|Y^m_{n_{k(l)}}|\, 1_{\{|Y^m_{n_{k(l)}}| > M\}}\Big] \stackrel{l,M\to\infty}{\longrightarrow} 0$.
We estimate the third term in (1) with the help of Fatou's lemma:

$$E^Q\big[\widetilde{Y}^{2m}\big] \leq \liminf_{l\to\infty} E^Q\big[Y^{2m}_{n_{k(l)}}\big] = \alpha_{2m}.$$

Note that each $Y^{2m}_{n_{k(l)}}$ is non-negative thanks to the even exponent $2m$ so that Fatou's lemma is valid in this case. This implies that $\widetilde{Y}^{2m} \in L^1$ and therefore, by Jensen's inequality, $\widetilde{Y}^m \in L^1$.

Consequently, $E^Q\left[|\widetilde{Y}^m|\,1_{\{|\widetilde{Y}^m|>M\}}\right] \xrightarrow{M\to\infty} 0$. By (1) the convergence of the moments follows, i.e. $E^Q\left[Y^m_{n_{k(l)}}\right] \to E^Q\left[\widetilde{Y}^m\right]$ as $l\to\infty$. Since, by assumption, $E^Q\left[Y^m_{n_{k(l)}}\right] \to \alpha_m$ as $l\to\infty$, we have that the mth moment of $\widetilde{P}$ is α_m for all $m \geq 1$. Since P is the unique measure that satisfies the latter equality for all $m \in \mathbb{N}$, we get $P = \widetilde{P}$.

Remark:

Now let us give a more detailed explanation of the remark after Exercise 5.3. For each $a \in [-1,1]$, we define a measure μ_a on $\mathbb{R}$ via

$$\mu_a(dx) := \frac{1}{\sqrt{2\pi}x}\exp\left(-\frac{(\log x)^2}{2}\right)(1 + a\sin(2\pi\log x))\mathbf{1}_{(0,\infty)}(x)\,dx.$$

First let us verify that μ_a actually defines a probability measure for all $a \in [-1,1]$. Indeed, using the substitution $y = \log(x)$ we have

$$\begin{aligned}\int_{\mathbb{R}}\mu_a(dx) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{1}{x}\exp\left(-\frac{(\log x)^2}{2}\right)(1 + a\sin(2\pi\log x))\,dx\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \exp\left(-\frac{y^2}{2}\right)(1 + a\sin(2\pi y))\,dy\\ &= \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \exp\left(-\frac{y^2}{2}\right)dy}_{=1} + \underbrace{\frac{a}{\sqrt{2\pi}}\int_{-\infty}^\infty \exp\left(-\frac{y^2}{2}\right)\sin(2\pi y)\,dy}_{=0,\text{ since }\sin\text{ is an odd function}} = 1.\end{aligned}$$

(For $a = 0$, the measure μ_0 is called the distribution of *standard log-normal* random variable, that is, μ_0 is the distribution of random variable $Y = \exp(X)$ with X a standard normal random variable.) We now claim that the kth moment $\int x^k\,\mu_a(dx)$ does *not* depend on a. Indeed, for any $a \in [-1.1]$, using the substitution $x = e^{k+u}$ we get:

$$\begin{aligned}\int_{\mathbb{R}} x^k\mu_a(dx) &= \int_0^\infty \frac{x^{k-1}}{\sqrt{2\pi}}\exp\left(-\frac{(\log x)^2}{2}\right)(1 + a\sin(2\pi\log x))\,dx\\ &= \int_{-\infty}^\infty \frac{\exp(k(k+u))}{\sqrt{2\pi}}\exp\left(-\frac{(k+u)^2}{2}\right)(1 + a\sin(2\pi(k+u)))\,du\\ &= \frac{e^{\frac{k^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{\frac{-u^2}{2}}(1 + a\sin(2\pi(k+u)))\,du\\ &= \underbrace{\frac{e^{\frac{k^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{\frac{-u^2}{2}}\,du}_{=e^{\frac{k^2}{2}}} + \underbrace{\frac{ae^{\frac{k^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{\frac{-u^2}{2}}\sin(2\pi(k+u))\,du}_{=0,\text{ since }\sin\text{ is odd}} = e^{\frac{k^2}{2}}.\end{aligned}$$

Clearly, this counterexample shows that in general a probability measure P can *not* be uniquely determined by its moments. Therefore the hypothesis regarding the uniqueness of P in the statements of Exercise 5.3 can not be removed.