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Probability Theory

Solution sheet 6

Solution 6.1 Let $S_n = \sum_{i=1}^n X_i$.

(a) Note that $\frac{1}{n^{1/\alpha}}(X_1 + \cdots + X_n) = n^{-1/\alpha} S_n$. Using that the random variables are i.i.d. and that the characteristic function is given by $\varphi_{X_1}(t) = \exp(-c|t|^\alpha)$ with $c > 0$,

$$\begin{aligned}\varphi_{\frac{S_n}{n^{1/\alpha}}}(t) &= \varphi_{S_n}(t/n^{1/\alpha}) = \prod_{i=1}^n \varphi_{X_i}(t/n^{1/\alpha}) = \varphi_{X_1}(t/n^{1/\alpha})^n \\ &= \left(e^{-c|t|^\alpha/n}\right)^n = e^{-c|t|^\alpha} = \varphi_{X_1}(t),\end{aligned}$$

showing that $\frac{1}{n^{1/\alpha}} S_n$ is distributed as X_1.

(b) Note that

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} = \frac{S_n}{n^{1/\alpha}} \frac{n^{1/\alpha}}{\sqrt{n}}.$$

By (a),

$$\varphi_{\frac{S_n}{n^{1/\alpha}} \frac{n^{1/\alpha}}{\sqrt{n}}}(t) = \varphi_{\frac{S_n}{n^{1/\alpha}}}\left(\frac{n^{1/\alpha}}{\sqrt{n}} t\right) = \varphi_{X_1}\left(\frac{n^{1/\alpha}}{\sqrt{n}} t\right).$$

Since $\alpha \in (0, 2)$,

$$\lim_{n\to\infty} \varphi_{\frac{S_n}{n^{1/\alpha}}}\left(\frac{n^{1/\alpha}}{\sqrt{n}} t\right) = \lim_{n\to\infty} \varphi_{X_1}\left(\frac{n^{1/\alpha}}{\sqrt{n}} t\right) = \lim_{n\to\infty} \exp(-c|n^{1/\alpha - 1/2} t|^\alpha) = \begin{cases} 1, & \text{if } t = 0, \\ 0, & \text{otherwise,} \end{cases}$$

which, since it is not continuous, is not the characteristic function of any distribution. Hence, by the contrapositive of (2.3.24) from the lecture notes,

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}}$$

does not converge in distribution.

Solution 6.2 Let $\mu_j := X_j \circ P$ be the distribution of X_j (cf. (1.2.15)). We consider the space $\left(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)\right)$, and let $\nu = \times_{j=1}^n \mu_j$ be the product measure of μ_j's $(j = 1, \ldots, n)$ on $\left(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)\right)$, which is the unique probability measure on $\left(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)\right)$ such that for $B_j \in \mathcal{B}(\mathbb{R})$,

$$\mu(B_1 \times \ldots \times B_n) = \prod_{j=1}^n \mu_j(B_j). \tag{1}$$

For more information about the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ and product measure $\times_{j=1}^n \mu_j$ we refer you to Chapter 14 of the book "Probability theory, a comprehensive course" by A. Klenke (English version), in particular Theorems 14.8 and 14.14 therein.
Let $\mu = (X_1, \ldots, X_n) \circ P$ be the image measure of the random vector $(X_1, \ldots, X_n)$ (cf. (1.2.15) again), which is a distribution on $\left(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)\right)$. From the equation (1) we can immediately see that $X_1, \ldots, X_n$ are independent if and only if $\mu = \nu = \times_{j=1}^n \mu_j$. Indeed, the independence condition $\prod_{j=1}^n P[X_j \in B_j] = P[X_1 \in B_1, \ldots, X_n \in B_n] = P[(X_1, \ldots, X_n) \in B_1 \times \ldots \times B_n] = \mu(B_1 \times \ldots \times B_n)$

means exactly that $\mu(B_1 \times \ldots \times B_n) = \prod_{j=1}^n \mu_j(B_j)$ for all $B_j \in \mathcal{B}(\mathbb{R})$, $j = 1, \ldots, n$. Since $\nu = \times_{j=1}^n \mu_j$ is the unique probability measure satisfies this property (as we have mentioned above) we must have $\mu = \nu$.

Now we can use Fubini's theorem (see e.g. Theorem 14.16 in the book "Probability theory, a comprehensive course") to obtain that ν has the characteristic function

$$\begin{aligned}\phi_\nu(\xi_1, \xi_2, \ldots, \xi_n) &= \int_{\mathbb{R}^n} e^{i\xi_1 x_1 + \ldots + i\xi_n x_n} \mu_1(\mathrm{d}x_1) \ldots \mu_n(\mathrm{d}x_n) \\ &= \prod_{j=1}^n \int e^{i\xi_j x_j} \mu_j(\mathrm{d}x_j) = \prod_{j=1}^n E\left[e^{i\xi_j X_j}\right].\end{aligned}$$

Hence, we have proven that:

$$X_1, \ldots, X_n \text{ are independent } \iff \mu = \nu \iff \phi_\mu = \phi_\nu,$$

where the last equivalence follows from the fact that characteristic functions also uniquely determine distributions on $\big(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)\big)$, see the hints given after Exercise 6.2. Now our claim follows by noting that the characteristic function of the vector $(X_1, \ldots, X_n)$ is given by

$$\phi_\mu(\xi_1, \ldots, \xi_n) = E\left[\exp\left\{i \sum_{j=1}^n \xi_j X_j\right\}\right],$$

and (as we have shown) that

$$\phi_\nu(\xi_1, \xi_2, \ldots, \xi_n) = \prod_{j=1}^n E\left[e^{i\xi_j X_j}\right].$$

Solution 6.3

(a) Let F_Z be the distribution function of Z. Let $a \in \mathbb{R}$ be a continuity point of F_Z and $\varepsilon > 0$. Since F_Z is non-decreasing, F_Z has at most countably many discontinuities. Therefore we can find a decreasing sequence $\delta_k > 0$ tending to zero, such that F_Z is continuous in $a + \delta_k$ for all $k \in \mathbb{N}$.

Now let $k \in \mathbb{N}$. By assumption there is a $N \in \mathbb{N}$, such that

$$P[|Y_n| > \delta_k] \leq \varepsilon, \quad n \geq N.$$

Furthermore, for $n \geq N$,

$$\begin{aligned}P[Z_n + Y_n \leq a] &\leq P[\{Z_n \leq a + \delta_k\} \cup \{|Y_n| > \delta_k\}] \\ &\leq P[Z_n \leq a + \delta_k] + \varepsilon\end{aligned}$$

and thus

$$\limsup_{n\to\infty} P[Z_n + Y_n \leq a] \leq \limsup_{n\to\infty} P[Z_n \leq a + \delta_k] + \varepsilon = F_Z(a + \delta_k) + \varepsilon.$$

Taking first the limit $k \to \infty$, using that a is a continuity point of F_z, and then $\varepsilon \to 0$,

$$\limsup_{n\to\infty} P[Z_n + Y_n \leq a] \leq F_Z(a).$$

A similar argument shows that

$$\liminf_{n\to\infty} P[Z_n + Y_n \leq a] \geq F_Z(a),$$

and thus the claim follows.

(b) For $u, \epsilon > 0$, since $\frac{N_n}{a_n}$ converges to 1 in probability, we have

$$P\left[\left|\frac{S_{N_n}}{\sigma\sqrt{a_n}} - \frac{S_{a_n}}{\sigma\sqrt{a_n}}\right| > u\right] = \underbrace{P\left[\left|\frac{S_{N_n}}{\sigma\sqrt{a_n}} - \frac{S_{a_n}}{\sigma\sqrt{a_n}}\right| > u, \left|\frac{N_n}{a_n} - 1\right| > \epsilon\right]}_{\leq \epsilon \text{ for n large enough}}$$
$$+ \underbrace{P\left[\left|\frac{S_{N_n}}{\sigma\sqrt{a_n}} - \frac{S_{a_n}}{\sigma\sqrt{a_n}}\right| > u, \left|\frac{N_n}{a_n} - 1\right| \leq \epsilon\right]}_{(*)}.$$

For the second term, we have (using Kolmogorov's inequality (1.4.4))

$$\begin{aligned}
(*) &\leq P\left[\max_{(1-\epsilon)a_n \leq k \leq (1+\epsilon)a_n}\left|\frac{S_k}{\sigma\sqrt{a_n}} - \frac{S_{a_n}}{\sigma\sqrt{a_n}}\right| > u\right] \\
&= P\left[\max_{(1-\epsilon)a_n \leq k \leq (1+\epsilon)a_n}|S_k - S_{a_n}| > u\sigma\sqrt{a_n}\right] \\
&\leq P\left[\max_{(1-\epsilon)a_n \leq k < a_n}|S_k - S_{a_n}| > u\sigma\sqrt{a_n}\right] \\
&\quad + P\left[\max_{a_n < k \leq (1+\epsilon)a_n}|S_k - S_{a_n}| > u\sigma\sqrt{a_n}\right] \\
&\leq \frac{1}{u^2\sigma^2 a_n}\sum_{\lceil(1-\epsilon)a_n\rceil}^{a_n}\mathrm{Var}(X_i) + \frac{1}{u^2\sigma^2 a_n}\sum_{a_n+1}^{\lfloor(1+\epsilon)a_n\rfloor}\mathrm{Var}(X_i) \\
&= \frac{1}{u^2\sigma^2 a_n}\sum_{\lceil(1-\epsilon)a_n\rceil}^{\lfloor(1+\epsilon)a_n)\rfloor}\mathrm{Var}(X_i) \\
&\leq \frac{1}{u^2 a_n}\big(\lfloor(1+\epsilon)a_n\rfloor - \lceil(1-\epsilon)a_n\rceil + 1\big) \leq \frac{2\epsilon}{u^2} + \frac{2}{a_n u^2} \xrightarrow{n\to\infty} \frac{2\epsilon}{u^2},
\end{aligned}$$

since $a_n \to \infty$ as $n \to \infty$. Here we set $\lceil x\rceil := \inf\{a \in \mathbb{N} : a \geq x\}$ and $\lfloor x\rfloor := \sup\{a \in \mathbb{N} : a \leq x\}$ for $x \geq 0$. Since $\epsilon > 0$ was arbitrary, we obtain that

$$\frac{S_{N_n}}{\sigma\sqrt{a_n}} - \frac{S_{a_n}}{\sigma\sqrt{a_n}} \xrightarrow{P} 0.$$

On the other hand, by the central limit theorem,

$$\frac{S_{a_n}}{\sigma\sqrt{a_n}} \text{ converges to } \mathcal{N}(0,1), \text{ as } n \to \infty, \text{ since } a_n \xrightarrow{n\to\infty} \infty.$$

Thus, we obtain the result by (a).