Probability Theory

Solution sheet 7

Solution 7.1 Since Z is constant on each the sets $A_0 = \{X + Y = 0\}$ and $A_1 = \{X + Y \geq 1\}$, we know that $\mathcal{G}$ is generated by this partition. Thus,

$$E[X|\mathcal{G}](\omega) = \alpha_i = \frac{E[X1_{A_i}]}{P(A_i)}, \quad \text{for } \omega \in A_i.$$

On A_0, X is identically 0, so $E[X1_{A_0}] = 0$ and $\alpha_0 = 0$. On the other hand, $X1_{A_1} = 1_{\{X=1\}}1_{\{X+Y\geq 1\}} = 1_{\{X=1\}}$, so it follows that

$$\alpha_1 = \frac{p}{P(A_1)} = \frac{p}{1-(1-p)^2} = \frac{1}{2-p}.$$

Hence, the conditional expectation can be expressed as

$$E[X|\mathcal{G}] = \frac{1}{2-p}1_{\{X+Y\geq 1\}}.$$

By symmetry, $E[Y|\mathcal{G}]$ is given by the same expression, whence we conclude that $E[X|\mathcal{G}] = E[Y|\mathcal{G}]$. Since a non-constant random variable cannot be independent from itself, the two random variables $E[X|\mathcal{G}]$ and $E[Y|\mathcal{G}]$ are not independent.

Solution 7.2

(a) The joint density of X and Y is the function that is constant and equals 2 on the given triangle, and zero outside. Clearly, we have $0 \leq Y/X \leq 1$, P-almost surely. Furthermore, for $t \in [0,1]$, we have

$$P[Y/X \leq t] = P[Y \leq tX] = \int_0^1 \int_0^{tx} 2\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 2tx\,\mathrm{d}x = t.$$

Thus Y/X has the uniform distribution on the interval $[0,1]$.

(b) Let $t_1, t_2 \in (0,1)$. Then we have

$$\begin{aligned} P[Y/X \leq t_1, X \leq t_2] = P[Y \leq t_1X, X \leq t_2] &= \int_0^{t_2}\int_0^{t_1x} 2\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^{t_2} 2t_1x\,\mathrm{d}x = t_1t_2^2 = P[Y/X \leq t_1]P[X \leq t_2]. \end{aligned}$$

This equality holds also trivially for $t_1 \notin (0,1)$ or $t_2 \notin (0,1)$. Thus Y/X and X are independent, since by (1.3.11) of the lecture notes, the distribution of $(Y/X, X)$ equals the product of the distributions of Y/X and X.

(c) Using the properties of the conditional expectation from the lecture, we have

$$E[Y|X] = E[(Y/X)X|X] \overset{(*)}{=} XE[Y/X|X] \overset{(**)}{=} XE[Y/X] \overset{(***)}{=} X/2.$$

- $(*)$ X is $\sigma(X)$-measurable.
- $(**)$ Y/X is independent from $\sigma(X)$ by part (b).

(∗∗∗) By (a) Y/X is uniformly distributed on $[0,1]$ and the expectation of the uniform distribution on $[0,1]$ is $1/2$.

Solution 7.3

We use the Lindeberg-Feller theorem (Theorem 2.24, p. 71 in lecture notes). First We define a triangular array of random variables

$$Y_{n,i} = \frac{X_i - E[X_i]}{\sqrt{\sum_{j=1}^n \operatorname{Var}(X_j)}}, \quad i = 1, \dots, n$$

(For the finitely many n where possibly $\sum_{j=1}^n \operatorname{Var}(X_j) = 0$, we set $Y_{n,i} \equiv 0$). Then it follows that

$$\sum_{i=1}^n E[Y_{n,i}^2] \xrightarrow{n\to\infty} 1.$$

More precisely, except for the finitely many n mentioned above,

$$\sum_{i=1}^n E[Y_{n,i}^2] = \sum_{i=1}^n \frac{E[(X_i - E[X_i])^2]}{\sum_{j=1}^n \operatorname{Var}(X_j)} = \frac{\sum_{i=1}^n \operatorname{Var}(X_i)}{\sum_{j=1}^n \operatorname{Var}(X_j)} = 1,$$

which justifies the first condition (2.4.7) in Theorem 2.24 with $\sigma^2 = 1$.

We now verify the second condition (2.4.8) in Theorem 2.24. For $\epsilon > 0$ we take $n_0 \in \mathbb{N}$ such that

$$\sum_{j=1}^n \operatorname{Var}(X_j) \geq \frac{(2M)^2}{\epsilon^2}, \quad \forall n \geq n_0,$$

which exists since $\sum \operatorname{Var}(X_j) = \infty$. Then by the fact that $|X_i|$ are uniformly bounded by M, one has

$$|Y_{n,i}| = \left| \frac{X_i - E[X_i]}{\sqrt{\sum_{j=1}^n \operatorname{Var}(X_j)}} \right| \leq \frac{2M}{2M/\epsilon} \leq \epsilon$$

for $n \geq n_0$. Hence,

$$1_{\{|Y_{n,i}|>\epsilon\}} \equiv 0, \ \forall n \geq n_0, \forall i \leq n,$$

and consequently

$$\lim_{n\to\infty} \sum_{i=1}^n E\left[Y_{n,i}^2 1_{\{|Y_{n,i}|>\epsilon\}}\right] = 0.$$

Therefore all the conditions of Theorem 2.24 are fulfilled, whence (see (2.4.9))

$$\sum_{i=1}^n Y_{n,i} \xrightarrow{n\to\infty} \mathcal{N}(0,1) \text{ in distribution.}$$

On the other hand we can rewrite $\sum_{i=1}^n Y_{n,i}$ as

$$\sum_{i=1}^n Y_{n,i} = \frac{\sum_{i=1}^n (X_i - E[X_i])}{\sqrt{\sum_{j=1}^n \operatorname{Var}(X_j)}},$$

and the claim follows with

$$a_n := \sqrt{\sum_{j=1}^n \operatorname{Var}(X_j)}, \qquad b_n := E\left[\sum_{j=1}^n X_j\right].$$