

# Probability Theory

## Solution sheet 9

**Solution 9.1** Because  $S, T$  are bounded, there exists some  $k \geq 0$ , such that  $S \leq T \leq k$   $P$ -almost surely. We then observe that  $X_S, X_T$  are integrable because both of them are dominated by the integrable random variable  $|X_0| + \dots + |X_k|$ .

Now let  $F \in \mathcal{F}_S$ . We define a sequence  $(C_n)_{n \geq 1}$  of non-negative, bounded random variables through

$$C_n(\omega) := 1_F(\omega) 1_{(S(\omega), T(\omega)]}(n), \quad \omega \in \Omega, n \geq 1.$$

Because  $\{T \leq n-1\} \in \mathcal{F}_{n-1}$  and  $F \cap \{S \leq n-1\} \in \mathcal{F}_{n-1}$ , one has that

$$C_n = 1_F 1_{\{S < n\}} 1_{\{T \geq n\}} = 1_{F \cap \{S \leq n-1\}} 1_{\{T \leq n-1\}^c}$$

is  $\mathcal{F}_{n-1}$ -measurable. This implies that  $(C_n)_{n \geq 1}$  is predictable.

By Theorem 3.22, p.93 of the lecture notes, it follows that  $C \cdot X$  is a submartingale (with  $(C \cdot X)_0 = 0$ ). Hence it follows that

$$0 \leq E[(C \cdot X)_k] = E \left[ \sum_{n=1}^k C_n (X_n - X_{n-1}) \right] = E [(X_T - X_S) 1_F].$$

Because  $F \in \mathcal{F}_S$  is arbitrary, one has that  $E[X_T | \mathcal{F}_S] \geq X_S$ ,  $P$ -a.s..

**Solution 9.2** First we show that an  $\mathcal{F}_n$ -stopping time  $S$  can be defined equivalently by the conditions  $\{S = n\} \in \mathcal{F}_n$  or  $\{S \leq n\} \in \mathcal{F}_n$ .

If  $\{S = n\} \in \mathcal{F}_n$  for all  $n \geq 0$ , then for all  $0 \leq k \leq n$ ,  $\{S = k\} \in \mathcal{F}_k \subseteq \mathcal{F}_n$ , which implies that  $\{S \leq n\} = \bigcup_{k=0}^n \{S = k\} \in \mathcal{F}_n$ . On the other hand,  $\{S \leq n\} \in \mathcal{F}_n$  for all  $n \geq 0$  implies that

- $\{S = 0\} \in \mathcal{F}_0$ ;
- $\{S \leq n-1\} \in \mathcal{F}_{n-1} \subseteq \mathcal{F}_n$  for all  $n \geq 1$ . Hence one knows that for all  $n \geq 1$ ,  $\{S = n\} = \{S \leq n\} \setminus \{S \leq n-1\} \in \mathcal{F}_n$ .

From now on we will use the one most convenient for our purpose in the following.

- (a) In general,  $S-1$  need not be a stopping time. Intuitively, this is because to know whether  $S-1$  has ‘happened’ by time  $n$ , information about time  $n+1$  is needed. A counter example can be constructed as follows:

Let  $\Omega = \{0, 1\}$ ,  $\mathcal{F}_0 = \{\emptyset, \Omega\}$  and  $\mathcal{F}_n = \{\emptyset, \{0\}, \{1\}, \Omega\}$  for  $n \geq 1$ . We define

$$S = 1_{\{1\}} + 2 \cdot 1_{\{0\}}.$$

Then  $\{S \leq 0\} = \emptyset \in \mathcal{F}_0$ ,  $\{S \leq 1\} = \{1\} \in \mathcal{F}_1$ , and  $\{S \leq k\} = \Omega \in \mathcal{F}_k$  for all  $k \geq 2$ . Thus,  $S$  is a  $\mathcal{F}_n$ -stopping time. However,  $\{S-1 \leq 0\} = \{S \leq 1\} = \{1\} \notin \mathcal{F}_0$ , so  $S-1$  is not a stopping time.

- (b)  $S+1$  is a stopping time, since for any  $n \geq 0$ ,

$$\{S+1 \leq n\} = \{S \leq n-1\} \in \mathcal{F}_{n-1} \subseteq \mathcal{F}_n.$$

- (c)  $S \wedge T$  is a stopping time, since for any  $n \geq 0$ ,

$$\{S \wedge T \leq n\} = \{S \leq n\} \cup \{T \leq n\} \in \mathcal{F}_n.$$

(d)  $S \vee T$  is a stopping time, since for any  $n \geq 0$ ,

$$\{S \vee T \leq n\} = \{S \leq n\} \cap \{T \leq n\} \in \mathcal{F}_n.$$

(e)  $S + T$  is a stopping time, since for any  $n \geq 0$ ,

$$\{S + T = n\} = \bigcup_{k=0}^n \underbrace{\{S = k\}}_{\in \mathcal{F}_k \subseteq \mathcal{F}_n} \cap \underbrace{\{T = n - k\}}_{\in \mathcal{F}_{n-k} \subseteq \mathcal{F}_n} \in \mathcal{F}_n.$$

**Solution 9.3** As in the hint, we first check that  $X_n$  is an  $\mathcal{F}_n$ -submartingale. Clearly,  $X_n$  is  $\mathcal{F}_n$ -adapted. Furthermore,  $X_n$  is bounded for all  $n$ , so it is integrable. Finally,  $1_{\{T \leq n+1\}} \geq 1_{\{T \leq n\}}$  for every  $n \geq 0$ , since  $\{T \leq n\} \subseteq \{T \leq n+1\}$ . Due to this, and by the monotonicity property of conditional expectation, we obtain

$$E[X_{n+1} | \mathcal{F}_n] = E[1_{\{T \leq n+1\}} | \mathcal{F}_n] \geq E[1_{\{T \leq n\}} | \mathcal{F}_n] = 1_{\{T \leq n\}} = X_n \quad P\text{-a.s.}$$

Hence,  $X_n$  is an  $\mathcal{F}_n$ -submartingale, so the Doob decomposition (unique up to  $P$ -nullsets) must exist. In other words, there exists a martingale  $M_n$ ,  $n \geq 0$ , and a predictable, non-decreasing process  $A_n$ , with  $A_0 = 0$ , such that

$$X_n = M_n + A_n, \quad n \geq 0.$$

To find  $M_n$  and  $A_n$ , we follow the proof of existence of this decomposition, see Proposition 3.19. For our  $X_n$ , we have for  $k \geq 0$ :

$$\begin{aligned} E[X_k - X_{k-1} | \mathcal{F}_{k-1}] &= E[1_{\{T \leq k\}} - 1_{\{T \leq k-1\}} | \mathcal{F}_{k-1}] \\ &= E[1_{\{T=k\}} | \mathcal{F}_{k-1}] \\ &= E[1_{\{Y_k=1\}} 1_{\{T > k-1\}} | \mathcal{F}_{k-1}] \\ &= 1_{\{T > k-1\}} E[1_{\{Y_k=1\}} | \mathcal{F}_{k-1}] \\ &= 1_{\{T > k-1\}} P[Y_k = 1] \\ &= p 1_{\{T > k-1\}} (= A_k - A_{k-1}) \quad P\text{-a.s.}, \end{aligned} \tag{1}$$

since  $Y$  is independent of  $\mathcal{F}_{k-1}$ . Thus, we define

$$A_n := \sum_{k=1}^n p 1_{\{T > k-1\}} = p \cdot (T \wedge n), \quad n \geq 0, \tag{2}$$

and we have that

$$M_n = X_n - A_n = 1_{\{T \leq n\}} - p \cdot (T \wedge n), \quad n \geq 0,$$

is a  $\mathcal{F}_n$ -martingale, since one can verify with equations (1) and (2) that

$$E[X_{n+1} - A_{n+1} | \mathcal{F}_n] = X_n - A_n \quad P\text{-a.s.}$$

Furthermore,  $A_n$  is non-decreasing,  $A_0 = 0$  and  $A_n$  is predictable, since each indicator function in the sum in equation (2) is  $\mathcal{F}_{n-1}$ -measurable. Thus, the Doob decomposition is

$$X_n = M_n + A_n = \left(1_{\{T \leq n\}} - p \cdot (T \wedge n)\right) + p \cdot (T \wedge n), \quad n \geq 0.$$