Coordinator
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Probability Theory

Solution sheet 10

Solution 10.1 From (3.5.3) we calculate m:

$$m = 2p \begin{cases} < 1 & \text{if } p \in [0, \frac{1}{2}), \\ = 1 & \text{if } p = \frac{1}{2}, \\ > 1 & \text{if } p \in (\frac{1}{2}, 1]. \end{cases}$$

Thus, if $p \in [0, \frac{1}{2})$, our Galton-Watson process is subcritical, if $p = \frac{1}{2}$ it is critical, and if $p \in (\frac{1}{2}, 1]$ it is supercritical.

For a subcritical Galton-Watson process, we have $P[Z_n = 0 \text{ eventually}] = 1$ by (3.5.7), p. 99, and by (3.5.10), p. 100 of the lecture notes also for a critical process. Hence,

$$\vartheta(p) = 0 \quad \forall p \in [0, 1/2].$$

In the supercritical case, we have, by (3.5.13), p. 101 of the lecture notes,

$$P[Z_n = 0 \text{ eventually}] = \varrho \in [0, 1),$$

where ϱ is the unique solution to $\varrho = \varphi(\varrho)$ in $[0, 1)$, and let X be a random variable with distribution ν, we have

$$\varphi(z) = E[z^X] = \sum_{k=0}^{2} P[X = k] z^k = (1-p)^2 + 2p(1-p)z + p^2 z^2,$$

(see (3.5.11), p. 100 of the lecture notes, and the explanations right below it). Solving the quadratic equation

$$\varphi(z) = \big((1-p) + pz\big)^2 = z$$

for z, we obtain the solutions $z = 1$ and $z = \frac{(1-p)^2}{p^2}$. Thus, the unique solution to $\varphi(\varrho) = \varrho$ in $[0, 1)$ is

$$\varrho = \frac{(1-p)^2}{p^2},$$

from which it follows that

$$\vartheta(p) = 1 - P[Z_n = 0 \text{ eventually}] = 1 - \varrho = 1 - \frac{(1-p)^2}{p^2} = \frac{2p-1}{p^2},$$

for $p \in (1/2, 1]$.

Solution 10.2

(a) Note that the situation of this exercise coincides with the so-called Polya's Urn (Exercise 8.2) with $s = w = 1$ and $t = 2$. Hence it follows immediately by using the same proof as for Exercise 8.2.

(b) Since $\left(\frac{Z_n}{n+1}\right)_{n \in \mathbb{N}}$ is a bounded martingale, it follows by the convergence theorem ((3.4.23), p.96 of the lecture notes) that it converges P-almost surely to a random variable X. Since $0 \le \frac{Z_n}{n+1} \le 1$, the convergence is also in L^1 by the dominated convergence theorem. To determine the distribution of X, we show that for each n, Z_n is uniformly distributed on

$\{1, \ldots, n\}$, using induction. This can be guessed by working out explicitly the distribution of Z_n for small n, or from the hint. We have

$$P[Z_1 = k] = \delta_1(k), \text{ since } Z_1 = 1.$$

Now assume that $P[Z_n = k] = \frac{1}{n}$, for $k = 1, \ldots, n$. Then, for all $k = 2, \ldots, n$, we have

$$\begin{aligned} P[Z_{n+1} = k] &= P[Z_{n+1} = k | Z_n = k] \frac{1}{n} + P[Z_{n+1} = k | Z_n = k-1] \frac{1}{n} \\ &= \left(1 - \frac{k}{n+1}\right) \frac{1}{n} + \frac{k-1}{n+1} \frac{1}{n} = \frac{1}{n+1}. \end{aligned}$$

For $k = n+1$, we have

$$P[Z_{n+1} = n+1] = P[Z_{n+1} = n+1 | Z_n = n] \frac{1}{n} = \frac{n}{n+1} \frac{1}{n} = \frac{1}{n+1}.$$

Similarly, one can deduce $P[Z_{n+1} = 1] = \frac{1}{n+1}$. Thus, we have

$$Z_n \overset{\mathrm{d}}{=} \frac{1}{n} \sum_{k=1}^{n} \delta_k \Rightarrow \frac{Z_n}{n+1} \overset{\mathrm{d}}{=} \frac{1}{n} \sum_{k=1}^{n} \delta_{\frac{k}{n+1}}.$$

Now the distribution on the right-hand side converges weakly to the Lebesgue measure λ on $[0,1]$, so we get that

$$\frac{Z_n}{n+1} \xrightarrow{\mathrm{d}} \widetilde{X}, \text{ where } \widetilde{X} \overset{\mathrm{d}}{=} \lambda.$$

Since P-almost sure convergence implies convergence in distribution (see (2.2.13), (2.2.14), p.50 of the lecture notes), it also holds that

$$\frac{Z_n}{n+1} \xrightarrow{\mathrm{d}} X.$$

Thus by uniqueness of the limit in distribution, we obtain

$$X \overset{\mathrm{d}}{=} \lambda.$$

Solution 10.3 Without loss of generality, we assume that $X_0 = 0$ or we just replace X_n by $X_n - X_0$.

Note that the hitting time T_A is an $\{\mathcal{F}_n\}_{n \geq 0}$-stopping time, for any $A \in \mathcal{B}(\mathbb{R})$, as (3.3.3) in Example 3.17, p. 89 of the lecture notes. Thus, from the optional stopping theorem ((3.4.15), p. 93 of the lecture notes), $X_{T_A \wedge n}$ is an $\{\mathcal{F}_n\}_{n \geq 0}$-martingale. If we let $A = [a, \infty)$ for $a > 0$, we furthermore have that

$$X_{T_{[a,\infty)} \wedge n} \leq a + M,$$

because of the bounded increments of X_n and $X_0 = 0$. This implies that we have

$$\sup_{n \geq 0} E\left[\left(X_{T_{[a,\infty)} \wedge n}\right)^+\right] \leq a + M < \infty.$$

Thus, by the martingale convergence theorem, (3.4.23), p. 96 of the lecture notes, the martingale $X_{T_{[a,\infty)} \wedge n}$ converges to some integrable random variable. But on the event $\{\sup_{n \geq 0} X_n < a\}$, we have $T_{[a,\infty)} = \infty$, so that $X_{T_{[a,\infty)} \wedge n} = X_n$ for all n. Thus on this event, X_n converges to a finite limit. From the definition of C, we obtain

$$P\left[C^c \cap \left\{\sup_{n \geq 0} X_n < a\right\}\right] = 0, \tag{1}$$

for all $a > 0$. Similarly by considering $-X_{T_{(-\infty,-a]}}$, or by symmetry, we can obtain that for all $a > 0$

$$P\left[C^c \cap \left\{\inf_{n\geq 0} X_n > -a\right\}\right] = 0. \tag{2}$$

Now by equations (1) and (2), we have

$$P\left[C^c \cap \left(\left\{\sup_{n\geq 0} X_n < a\right\} \cup \left\{\inf_{n\geq 0} X_n > -a\right\}\right)\right] = 0. \tag{3}$$

Taking the limit $a \to \infty$, and using the continuity property of measures, we get by definition of the event D

$$P[C^c \cap D^c] = 0. \tag{4}$$

Now the claim follows by taking the complement event in (4).
Remark: This exercise is the essential ingredient of the proof of the generalised version of the second Borel-Cantelli Lemma, see Theorems 5.31 and 5.32 in Durrett's book (pp. 204-205 in 4th online edition, pp. 239-240 in 3rd edition).