Coordinator
Chong Liu

# Probability Theory

## Solution sheet 11

**Solution 11.1** We define $X_0 := 0$, $X_n := \sum_{m=1}^{n} \left(1_{A_m} - P[A_m|\mathcal{F}_{m-1}]\right)$, $n \geq 1$. Then $X_n$ is an $\mathcal{F}_n$-martingale, since

$$E[X_{n+1} - X_n|\mathcal{F}_n] = E\left[1_{A_{n+1}} - P[A_{n+1}|\mathcal{F}_n]\middle|\mathcal{F}_n\right] = 0.$$

Furthermore $|X_{n+1} - X_n| \leq 2$. We apply the result of Exercise 10.3 to obtain $P[C \cup D] = 1$. Note that:

- $\sum_{n\geq 1} 1_{A_n} = \infty \iff \sum_{n\geq 1} P[A_n|\mathcal{F}_{n-1}] = \infty$ on $C$.
- $\sum_{n\geq 1} 1_{A_n} = \infty$ and $\sum_{n\geq 1} P[A_n|\mathcal{F}_{n-1}] = \infty$ on $D$.

Using that $P[C \cup D] = 1$, we get that for an event $N$ with $P[N] = 0$,

$$\left\{\sum_{n\geq 1} 1_{A_n} = \infty\right\} \cap N^c = \left\{\sum_{n\geq 1} P[A_n|\mathcal{F}_{n-1}] = \infty\right\} \cap N^c.$$

Finally, the claim follows since

$$\limsup_{n\to\infty} A_n = \left\{\sum_{n\geq 1} 1_{A_n} = \infty\right\}.$$

**Solution 11.2** Let $S_n = \sum_{m=1}^{n} X_m$ ($S_0 = 0$) be the simple random walk on $\mathbb{Z}^d$ such that $X_n$, $n \geq 1$ are i.i.d. $\mathbb{Z}^d$ valued random variable with $P[X_1 = e] = 1/(2d)$ for any $e \in \mathbb{R}^d$, $\|e\| = 1$ (i.e., $e = e_i$ or $e = -e_i$ for $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 on the $i$th entry, $i = 1, \ldots, d$). We can easily show that $S_n$ is an $\mathcal{F}_n$ martingale taking values in $\mathbb{Z}^d$ and $S_n^j$, the $j$th coordinate of $S_n$, is also an $\mathcal{F}_n$ real–valued martingale for each $j = 1, \ldots, d$.

(a) Consider the martingale

$$M_n = \sum_{i=1}^{d} S_n^i,$$

with bounded increments

$$M_n - M_{n-1} = \sum_{i=1}^{d} (S_n^i - S_{n-1}^i) \in \{-1, 1\}.$$

By Exercise 10.3, it thus holds that

$$P\left[\{\lim M_n \in \mathbb{Z} \text{ exists}\} \cup \{\liminf M_n = -\infty \text{ and } \limsup M_n = \infty\}\right] = 1.$$

However, since $|M_n - M_{n-1}| = 1$, $M_n$ cannot converge, from which it follows that $\limsup M_n = \infty$ $P$-a.s. and $\liminf M_n = -\infty$ $P$-a.s. This, in turn, implies that $S_n$ must exit the finite set $A$, since $\max_{(a_1,\ldots,a_d)\in A} \sum_{i=1}^{d} |a_i| < \infty$.

(b) If $x \notin A$, then $T_x = 0 \Rightarrow f(x + S_{T_x}) = f(x)$.

If $x \in A$, then

$$\begin{aligned} E[f(x+S_{T_x})] &= \sum_{\|e\|=1} E\left[f\left(x+S_{T_x}\right)\mathbf{1}_{S_1=e}\right] \\ &= \frac{1}{2d}\sum_{\|e\|=1} E\left[f\left(x+e+S_{T_{x+e}}\right)\right]. \end{aligned}$$

To see the last equality, we note that if $x+e \notin A$, then in view of the definition of $T_{x+e}$ (resp. $T_x$) we have $T_{x+e}=0$ (resp. $T_x=1$) and in this case it holds that $E\left[f\left(x+S_{T_x}\right)\mathbf{1}_{S_1=e}\right] = f(x+e)P[S_1=e] = \frac{f(x+e)}{2d} = E\left[f\left(x+e+S_{T_{x+e}}\right)\right]$. On the other hand, if $x+e\in A$, then we must have $T_x \geq 2$ and $T_{x+e}\geq 1$. In this case we have

$$\begin{aligned} E\left[f\left(x+S_{T_x}\right)\mathbf{1}_{S_1=e}\right] &= \sum_{n\geq 2} E\left[f\left(x+e+S_n-S_1\right)\mathbf{1}_{\{S_1=e\}\cap\{T_x=n\}}\right] \\ &= \sum_{n\geq 2} E\left[f\left(x+e+S_n-S_1\right)\mathbf{1}_{\{S_1=e\}\cap B_n}\right], \end{aligned}$$

where $B_n = \{\forall 2\leq m<n, x+e+(S_m-S_1)\in A; x+e+S_n-S_1\notin A\}$. But since $S_m - S_1 = \sum_{j=2}^m X_j$ is independent of $S_1 = X_1$ for all $m\geq 2$, the event $B_n$ is independent of $\{S_1=e\}$. Moreover, it is easy to see that $x+e+S_m-S_1 = x+e+\sum_{j=2}^m X_j$, $m\geq 1$ is the simple random walk started from $x+e$, which implies that $B_n = \{T_{x+e}=n-1\}$ for all $n\geq 2$ and $x+e+S_n-S_1 = x+e+S_{T_{x+e}}$ on $B_n$. Hence, we can deduce that

$$\begin{aligned} \sum_{n\geq 2} E\left[f\left(x+e+S_n-S_1\right)\mathbf{1}_{\{S_1=e\}\cap B_n}\right] &= \sum_{n\geq 2} E\left[f\left(x+e+S_n-S_1\right)\mathbf{1}_{B_n}\right]P[S_1=e] \\ &= \frac{1}{2d}E\left[f\left(x+e+S_{T_{x+e}}\right)\mathbf{1}_{\{T_{x+e}\geq 1\}}\right]. \end{aligned}$$

Now we complete our proof by writing $y = x+e$ for some $e$ with $\|e\|=1$.

(c) In case $x\notin A$ the statement is trivial.

If $x\in A$, then $S_{T_x\wedge 1} = S_1$, whence it follows that

$$E[f(x+S_{T_x})\mathbf{1}_{S_1=e}] = \frac{1}{2d}E[f(x+e+S_{T_{x+e}})] = \frac{1}{2d}g(x+e)$$

and

$$E[g(x+S_1)\mathbf{1}_{S_1=e}] = \frac{1}{2d}E[g(x+e)] = \frac{1}{2d}g(x+e).$$

This concludes the proof.

**Solution 11.3**

(a) Since $(M_n)_{n\geq 0}$ is adapted to the filtration $\mathcal{F}_n = \sigma(Y_0,\dots,Y_n)_{n\in\mathbb{N}}$ and, for $n\in\mathbb{N}$, $E[|M_n|] = E[Y_1]\cdots E[Y_n] = 1$, it holds that $M_n \in L^1$. Moreover,

$$\begin{aligned} E[M_{n+1}|\mathcal{F}_n] &= E[M_nY_{n+1}|\mathcal{F}_n] \\ {\scriptstyle (M_n\ \mathcal{F}_n\text{-meas.})}\ &= M_nE[Y_{n+1}|\mathcal{F}_n] \\ {\scriptstyle (Y_{n+1},\ \mathcal{F}_n\ \text{indep.})}\ &= M_nE[Y_{n+1}] = M_n. \end{aligned}$$

Hence, $(M_n)_{n\geq 0}$ is a $\mathcal{F}_n$-martingale with $\sup_{n\in\mathbb{N}} E[M_n] = 1 < \infty$ and, by the martingale convergence theorem ((3.4.23) in the lecture notes), $M_n$ converges to an integrable random variable $M_\infty$ $P$-a.s.

(b) From the Cauchy-Schwarz (or Hölder) inequality it follows that,

$$a_n = E[\sqrt{Y_n}] \leq E[Y_n]^{1/2} E[1]^{1/2} = 1.$$

Thus, since $Y_n \geq 0$ $P$-a.s. and therefore $\sqrt{Y_n} \geq 0$ $P$-a.s. and $E[Y_n] = 1$, it must hold that $a_n > 0$.

(c) We define $\hat{Y}_n = \sqrt{Y_n}/a_n$ and $\hat{M}_n = \hat{Y}_1\hat{Y}_2, \ldots, \hat{Y}_n$, $\hat{M}_0 = 1$. Then, from (a), the process $(\hat{M}_n)_{n\in\mathbb{N}}$ is also a non-negative martingale, converging to an integrable random variable $\hat{M}_\infty$. Furthermore $M_n \leq \hat{M}_n^2$ for $n \in \mathbb{N}$.

Suppose that $\prod_n a_n > 0$. Then we get,

$$E[\hat{M}_n^2] = (a_1 a_2 \cdots a_n)^{-2} \leq \left(\prod_n a_n\right)^{-2} < \infty.$$

Therefore, $\hat{M}_n \in L^2$, and by Doob's inequality (see Corollary 3.34),

$$E\left[\sup_{n\in\mathbb{N}} M_n\right] \leq E\left[\sup_{n\in\mathbb{N}} \hat{M}_n^2\right] \leq 4 \sup_{n\in\mathbb{N}} E[\hat{M}_n^2] < \infty.$$

From Lebesgue convergence theorem ($|M_n - M_\infty| \leq 2\sup_{n\in\mathbb{N}} M_n \in L^1$) we obtain convergence also in $L^1$.

(d) Suppose that $\prod_n a_n = 0$. Then, from the $P$-a.s. convergence of $\hat{M}_n = \sqrt{M_n}/(\prod_{i=1}^n a_i)$ to an a.s. finite random variable $\hat{M}_\infty \in L^1$, it follows that $M_\infty = 0$ a.s.